Contraction, propagation and bisection on a validated simulation of ODE

Julien Alexandre dit Sandretto
Alexandre Chapoutot
Recall on validated simulation

Contraction on a validated simulation

Propagation

Experimentations

Discussion
Recall on validated simulation

Initial value problem

\[ \dot{y} = f(y, p, t) \text{ with } y(0) \in [y_0] \text{ and } p \in [p] . \]

Classical approach

Lohner 2-steps method:

1. Find \([\tilde{y}_i]\) and \(h_i\) with Picard-Lindelöf operator and Banach’s theorem

2. Compute \([y_{i+1}]\) with a validated integration scheme: Taylor (Vnode-LP, CAPD) or Runge-Kutta (DynIbex)
Recall on validated simulation

Validated simulation ⇒ Two lists of boxes

\[ S = \{ [y_0], \ldots, [y_i], \ldots, [y_N] \} \] and \[ \tilde{S} = \{ [\tilde{y}_0], \ldots, [\tilde{y}_i], \ldots, [\tilde{y}_N] \}, \]

such that, for all \( p \) and \( y_0 \),

- \( y(t_i) \in [y_i] \) with \( t_i \in \{0, t_1, \ldots, t_N \} \)
- \( y(t) \in [\tilde{y}_i] \) for all \( t \in [t_i, t_{i+1}] \)

red: \( \tilde{S} \)

black: \( S \)
Recall on validated simulation

Contractors

An interval contractor (Ctc)

- $Ctc([x]) \subset [x]$ (contractance property)
- $Ctc([x]) \cap \mathcal{X} = [x] \cap \mathcal{X}$ (completeness property)

A contractor associated to a constraint is a contractor associated to the set $\mathcal{X}$ of all $x$ which satisfy the constraint.

Contractors in simulation

- Picard operator is a contractor
  $$Ctc_{\text{picard}}([\tilde{y}_j]) := [\tilde{y}_j] \cap [y] + \int_0^h f([\tilde{y}_j]) ds$$

- Validated integration scheme is a contractor
  $$Ctc_{\text{integration}}([y_{j+1}]) := [y_{j+1}] \cap [y] + \int_0^h f([y_j]) ds$$

(Enclosed by validated Taylor or RK)
Recall on validated simulation

Contractors based on tricks

Monotonicity contractor

If $0 \notin f([\tilde{y}_j])$, monotonicity allows $[\tilde{y}_j] = [y_j] \cup [y_{j+1}]$ (and recomputation of $[y_{j+1}]$ with new remainder and fixed point...)

Slicing contractor

If $[y_{j+1}] \supset [y_j]$ (possible attractor in $[y_j]$), slicing of $[y_j]$, integration of slices, and $[y_{j+1}] = \text{union of results}$...
Constraint Satisfaction Differential Problems

Problem

If information is given on \( y(t) \) ⇒ how take it into account ?

Information on \( y(t) \)

Given under the form of a temporal set constraints: \( y(t^*) \in A \)

Note: \( A \) can be the result of a contraction of \( y(t^*) \)

w.r.t. a constraint (e.g. \( y(t^*)^2 - 3 \cos(y(t^*)) < 0 \) or \( y(t^*) \notin B \))

For us, a CSDP is a set of temporal set constraints:

\[
\{ y(t_1) \in A_1, \ldots, y(t_z) \in A_z \}
\]

Our Approach

Contraction and propagation of already computed simulation:

avoids costly Lohner first step
Constraint Satisfaction Differential Problems

Problem

If information is given on $y(t) \Rightarrow$ how take it into account?

Information on $y(t)$

Given under the form of a temporal set constraints: $y(t^*) \in \mathcal{A}$

Note: $\mathcal{A}$ can be the result of a contraction of $y(t^*)$

w.r.t. a constraint (e.g. $y(t^*)^2 - 3 \cos(y(t^*)) < 0$ or $y(t^*) \notin \mathcal{B}$)

For us, a CSDP is a set of temporal set constraints:

$$\{y(t_1) \in \mathcal{A}_1, \ldots, y(t_z) \in \mathcal{A}_z\}$$

Our Approach

Contraction and propagation of already computed simulation:

avoids costly Lohner first step
Constraint Satisfaction Differential Problems

Problem

If information is given on $y(t) \Rightarrow$ how take it into account?

Information on $y(t)$

Given under the form of a temporal set constraints: $y(t^*) \in A$

Note: $A$ can be the result of a contraction of $y(t^*)$

w.r.t. a constraint (e.g. $y(t^*)^2 - 3 \cos(y(t^*)) < 0$ or $y(t^*) \notin B$)

For us, a CSDP is a set of temporal set constraints:

\[ \{ y(t_1) \in A_1, \ldots, y(t_z) \in A_z \} \]

Our Approach

Contraction and propagation of already computed simulation:

avoids costly Lohner first step
How take a temporal set constraint into account?

1-Add a new integration step

Simulation becomes: \( S = \{ [y_0], \ldots, [y_i], [y_k], [y_{i+1}], \ldots, [y_N] \} \)

s.t. \( t_k = t^* \) and \( y(t_k) \in [y_k] \), and same for \( \tilde{S} \)

\( \Rightarrow \) without Lohner first step because \( y(t_k) \in [\tilde{y}_i] \) and \( h = t_k - t_i \)

2-Contraction

Apply the basic contractor \([y_k] := [y_k] \cap A\)
How take a temporal set constraint into account?

\[ y(t^*) \in \mathcal{A} \]

1-Add a new integration step

Simulation becomes: \( S = \{ [y_0], \ldots, [y_i], [y_k], [y_{i+1}], \ldots, [y_N] \} \)

S.t. \( t_k = t^* \) and \( y(t_k) \in [y_k] \), and same for \( \tilde{S} \)

\( \Rightarrow \) without Lohner first step because \( y(t_k) \in [\tilde{y}_i] \) and \( h = t_k - t_i \)

2-Contraction

Apply the basic contractor \( [y_k] := [y_k] \cap \mathcal{A} \)
How take a temporal set constraint into account?

\[ y(t^*) \in \mathcal{A} \]

1-Add a new integration step

Simulation becomes: \( S = \{ [y_0], \ldots, [y_i], [y_k], [y_{i+1}], \ldots, [y_N] \} \)

s.t. \( t_k = t^* \) and \( y(t_k) \in [y_k] \), and same for \( \tilde{S} \)

\( \Rightarrow \) without Lohner first step because \( y(t_k) \in [\tilde{y}_i] \) and \( h = t_k - t_i \)

2-Contraction

Apply the basic contractor \([y_k] := [y_k] \cap \mathcal{A}\)
Remark on bisection

Of boxes

Bisection of $[A]$:
$[A_{left}] := Ctc_{left}([A])$ and $[A_{right}] := Ctc_{right}([A])$

Of states

Bisection of $y(t)$ at $t = t^*$:
$S_{left} = S$ and $S_{right} = S$, then $S_{left} : y(t^*) \in Ctc_{left}([y_k])$ and $S_{right} : y(t^*) \in Ctc_{right}([y_k])$
Remark on bisection

Of boxes

Bisection of $[A]$:

$[A_{left}] := Ctc_{left}([A])$ and $[A_{right}] := Ctc_{right}([A])$

Of states

Bisection of $y(t)$ at $t = t^*$:

$S_{left} = S$ and $S_{right} = S$, then $S_{left}: y(t^*) \in Ctc_{left}([y_k])$ and $S_{right}: y(t^*) \in Ctc_{right}([y_k])$
Propagation

Important to propagate \( y(t^*) \in A \) to \( y(t), \forall t \)

Forward propagation

From \( t^* \) to \( t_{\text{end}} \):

\[ C_{\text{tc_{picard}}} + C_{\text{tc_{integration}}} \]

Remark: Picard first to reduce Lagrange remainder of integration

Backward propagation

From \( t^* \) to \( t_0 \):

\[ C_{\text{tc_{picard}}} + C_{\text{tc_{integration}}} \]

with inverse function \(-f\) and past stepsize \( h = t_i - t_{i-1} \)

Fixed point

While sufficient improvement w.r.t. a given threshold:

From \( t_0 \) to \( t_{\text{end}} \):

Forward propagation

From \( t_{\text{end}} \) to \( t_0 \):

Backward propagation
Propagation

Important to propagate “\(y(t^*) \in \mathcal{A}\)” to \(y(t), \forall t\)

Forward propagation

From \(t^* \) to \(t_{end}\):

\[ Ctc_{picard} + Ctc_{integration} \]

Remark: Picard first to reduce Lagrange remainder of integration

Backward propagation

From \(t^* \) to \(t_0\):

\[ Ctc_{picard} + Ctc_{integration} \]

with inverse function \(-f\) and past stepsize \(h = t_i - t_{i-1}\)

Fixed point

While sufficient improvement w.r.t. a given threshold:

From \(t_0\) to \(t_{end}\):

Forward propagation

From \(t_{end}\) to \(t_0\):

Backward propagation
Propagation

Important to propagate \( y(t^*) \in A \) to \( y(t) \), \( \forall t \)

Forward propagation

From \( t^* \) to \( t_{\text{end}} \):

\[
Ctc_{\text{picard}} + Ctc_{\text{integration}}
\]

Remark: Picard first to reduce Lagrange remainder of integration

Backward propagation

From \( t^* \) to \( t_0 \):

\[
Ctc_{\text{picard}} + Ctc_{\text{integration}}
\]

with inverse function \(-f\) and past stepsize \( h = t_i - t_{i-1} \)

Fixed point

While sufficient improvement w.r.t. a given threshold:

From \( t_0 \) to \( t_{\text{end}} \):

Forward propagation

From \( t_{\text{end}} \) to \( t_0 \):

Backward propagation
Propagation

Important to propagate \( y(t^*) \in A \) to \( y(t), \forall t \)

Forward propagation

From \( t^* \) to \( t_{end} \):
\[
Ctc_{picard} + Ctc_{integration}
\]

Remark: Picard first to reduce Lagrange remainder of integration

Backward propagation

From \( t^* \) to \( t_0 \):
\[
Ctc_{picard} + Ctc_{integration}
\]

with inverse function \(-f\) and past stepsize \( h = t_i - t_{i-1} \)

Fixed point

While sufficient improvement w.r.t. a given threshold:

From \( t_0 \) to \( t_{end} \):
Forward propagation

From \( t_{end} \) to \( t_0 \):
Backward propagation
Experimentations

Example

Van Der Pol

\begin{align*}
   x' &= y \\
   y' &= 2.0(1.0 - x^2)y - x
\end{align*}

\[ x(0) \in [2.0, 2.2] \text{ and } y(0) \in [0.0, 0.1] \]

\[ t_{end} = 2.0 \]
Bisection then propagation

\[ x(0) = \{x(0)_{\text{left}}, x(0)_{\text{right}}\} \]

1 simulation then 2 propagations \( \text{vs} \) 3 simulations

Time: \( 2+2\times1 = 4 \) sec. \( \text{vs} \) \( 2\times3 = 6 \) sec.
Bisection then propagation

\( x(0) = \{ x(0)_{\text{left}}, x(0)_{\text{right}} \} \)

1 simulation then 2 propagations  vs  3 simulations

Time : 2+2*1 = 4 sec.  vs  2*3 = 6 sec.
Bisection then propagation

\[ x(0) = \{ x(0)_{left}, x(0)_{right} \} \]

1 simulation then 2 propagations  vs  3 simulations

Time: \( 2 + 2 \times 1 = 4 \text{ sec.} \) vs \( 2 \times 3 = 6 \text{ sec.} \)
Bisection then propagation

\[ x(0) = \{ x(0)_{left}, x(0)_{right} \} \]

1 simulation then 2 propagations vs 3 simulations

Time: \( 2 + 2 \times 1 = 4 \) sec. vs \( 2 \times 3 = 6 \) sec.
Experimentations

Contraction then propagation: Initial

$y(1.0) \in [1.58, 1.62]$ and $x(1.0) \in [-0.74, -0.69]$
Contraction then propagation : With Measure

$y(1.0) \in [1.58, 1.62]$ and $x(1.0) \in [-0.74, -0.69]$
Contraction then propagation: Contraction

\[ y(1.0) \in [1.58, 1.62] \text{ and } x(1.0) \in [-0.74, -0.69] \]
Contraction then propagation: Forward

\[ y(1.0) \in [1.58, 1.62] \text{ and } x(1.0) \in [-0.74, -0.69] \]
Contraction then propagation : Backward

\[ y(1.0) \in [1.58, 1.62] \text{ and } x(1.0) \in [-0.74, -0.69] \]
Experimentations

Contraction then propagation: Fixed Point

\[ y(1.0) \in [1.58, 1.62] \text{ and } x(1.0) \in [-0.74, -0.69] \]
Discussion

Remarks

▶ Propagation of a contraction on one state is close to a contraction on a tube [2]
▶ Easy to generalize to a contraction on parameters by changing $f(y, [p])$ by $f(y, Ctc([p]))$
▶ Easy to generalize to interval of time ($t = [t, \bar{t}]$ same job for $t$ and $\bar{t}$)
▶ Periodicity, limit cycle, etc.

Future work

Finalize implementation of CSDP in DynIBEX and applications!

Questions ?