Decision Making Under Twin Interval Uncertainty

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1. Decision Making: General Idea

- According to the decision theory:
  - a reasonable person should select an alternative a
  - for which an appropriate objective function $u(a)$ – called *utility* – attains its largest possible value.
- The utility function is usually selected in such a way that:
  - if for some action a, we know the probabilities p_i of different outcomes o_i,
  - then the utility of a is equal to the expected value of the utilities: $u(a) = \sum\limits_{i=1}^{n} p_i \cdot u(o_i)$.
- Such a utility function is determined uniquely modulo a linear transformation

$$u(a) \to u'(a) = k \cdot u(a) + \ell, \text{ where } k > 0.$$

2. Decision Making Under Interval Uncertainty

- For some actions, we have no information about the probabilities of different outcomes o_i.
- In this case, all we know about the expected utility $u(a)$ is that it is in the interval $[\underline{u}(a), \overline{u}(a)]$, where

$$\underline{u}(a) = \min_i u(o_i) \text{ and } \overline{u}(a) = \max_i u(o_i).$$

- To make decisions under such interval uncertainty, we must, in particular, we able to compare:
  - such actions with interval uncertainty with
  - actions for which we know the expected utility $u(a)$.
- Thus, we need to be able to assign, to each interval $[\underline{u}(a), \overline{u}(a)]$, an equivalent utility value $u(a)$.
- A way to assign such an equivalent utility value was proposed by a Nobel Prize winner Leo Hurwicz:

$$u(a) = \alpha \cdot \overline{u}(a) + (1 - \alpha) \cdot \underline{u}(a).$$

3. Decision Making Under Interval Uncertainty (cont-d)

- *Reminder:* $u(a) = \alpha \cdot \overline{u}(a) + (1 - \alpha) \cdot \underline{u}(a)$.
- Here, $\alpha \in [0, 1]$ describes the optimism level of the decision maker:
  - $\alpha = 1$ means that the decision maker only takes into account the best-case scenario,
  - $\alpha = 0$ means that only the worst-case scenario is taken into account, and
  - $\alpha \in (0, 1)$ means that both best-case and worst-case scenarios are taken into account.
- It turns out that the Hurwicz assignment is invariant relative to linear transformations of utility.
- It is actually the *only* invariant assignment.

4. Case of Twin Uncertainty

- In practice, sometimes, we do not know the exact values of $\underline{u}(a)$ and $\overline{u}(a)$.
- For example, we may only know the bounds on each of these bounds:
  - we know that $\underline{u}(a) \in [\underline{u}^-(a), \underline{u}^+(a)]$ and
  - we know that that $\overline{u}(a) \in [\overline{u}^-(a), \overline{u}^+(a)]$.
- Such a situation is known as a *twin interval*.
- How can we make decisions under such twin interval uncertainty?

5. Our Main Idea

- Our main idea is to use Hurwicz assignment several times.
- Specifically, for the lower bound $\underline{u}(a)$, all we know that it is in the interval $[\underline{u}^-(a), \underline{u}^+(a)]$.
- According to the Hurwicz assignment, this is equivalent to having $\underline{u}(a) = \alpha \cdot \underline{u}^+(a) + (1-\alpha) \cdot \underline{u}^-(a)$.
- Similar, for $[\overline{u}^-(a), \overline{u}^+(a)]$, we conclude that the upper bound is equivalent to $\overline{u}(a) = \alpha \cdot \overline{u}^+(a) + (1-\alpha) \cdot \overline{u}^-(a)$.
- Thus, the original twin interval is equivalent to the interval $[\underline{u}(a), \overline{u}(a)]$.
- For this interval, the Hurwicz assignment produces an equivalent value

$$u(a) = \alpha \cdot \overline{u}(a) + (1-\alpha) \cdot \underline{u}(a) =$$

$$\alpha^2 \cdot \overline{u}^+(a) + \alpha \cdot (1-\alpha) \cdot \overline{u}^-(a) + \alpha \cdot (1-\alpha) \cdot \underline{u}^+(a) + (1-\alpha)^2 \cdot \underline{u}^-(a).$$

6. This Idea Is Consistent

- Alternatively, we can consider the situation differently: namely, we consider the actual interval.
- The smallest possible interval – in terms of component-wise order – is $[\underline{u}^-(a), \overline{u}^-(a)]$.
- The largest possible interval is $[\underline{u}^+(a), \overline{u}^+(a)]$.
- For the smallest interval, Hurwicz's equivalent $u^-(a)$ is

$$u^-(a) = \alpha \cdot \overline{u}^-(a) + (1 - \alpha) \cdot \underline{u}^-(a).$$

- For the largest interval, the equivalent utility is $u^+(a) = \alpha \cdot \overline{u}^+(a) + (1 - \alpha) \cdot \underline{u}^+(a)$.
- Thus, possible values of utility form an interval $[u^-(a), u^+(a)]$.
- For this interval, the Hurwicz equivalent value is $\alpha \cdot u^+(a) + (1 - \alpha) \cdot u^-(a)$, same as before.

7. Towards Applications

- Some physical quantities we can measure directly.
- In many practical situations, we are interested in a quantity y which is difficult to measure directly.
- To estimate the values of such a quantity, a natural idea is:
  - find easier-to-measure quantities $x_1, \ldots, x_n$ related to y by a known dependence $y = f(x_1, \ldots, x_n)$,
  - and then use the results $\widetilde{x}_i$ of measuring x_i to compute the estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.
- Often, the only info that we have about each measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is the upper bound Δ_i:

$$|\Delta x_i| \le \Delta_i.$$

8. Case of Interval Uncertainty

- In this case, the only information that we have about the actual (unknown) value x_i is that

$$x_i \in [\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i].$$

- Usually, we do not know the dependence between x_i (and we do not even know if there is a dependence).
- The traditional interval approach to this situation is to conclude that y belongs to the range

$$\mathbf{y} \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i]\}.$$

- However, in reality, the range $[\underline{y}, \overline{y}]$ depends on the possible dependence between the variables x_i.

9. Need for Twin Interval Uncertainty

- In general, $\underline{y} = \inf\{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in S\}$ and $\overline{y} = \sup\{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in S\}$, where:
  - for every i,
  - the projection $\pi_i(S)$ on the i-th axis coincides with $[\underline{x}_i, \overline{x}_i]$.
- For different sets S, we have, in general, different values $\underline{y}$ and $\overline{y}$.
- It is therefore desirable to compute the ranges $[\underline{y}^-, \underline{y}^+]$ and $[\overline{y}^-, \overline{y}^+]$ of the corresponding values.
- In other words, it is desirable to compute the corresponding twin interval.
- Here, $\underline{y}^-$ and $\overline{y}^+$ are the endpoints of the range $\mathbf{y}$, which can computed by the usual interval techniques.
- So, the question is how to compute $\underline{y}^+$ and $\overline{y}^-$.

10. Our Main Result

- We consider a practically important case when terms quadratic in Δx_i can be safely ignored; in this case:

$$\Delta y = \widetilde{y} - y = \sum_{i=1}^{n} c_i \cdot \Delta x_i, \text{ where } c_i = \frac{\partial f}{\partial x_i}(\widetilde{x}_1, \ldots, \widetilde{x}_n).$$

- In turns out that in this case,

$$\overline{y}^{-} = \widetilde{y} + 2 \max_i(|c_i| \cdot \Delta_i) - \sum_{i=1}^{n}(|c_i| \cdot \Delta_i) \text{ and}$$

$$\underline{y}^{+} = \widetilde{y} - 2 \max_i(|c_i| \cdot \Delta_i) + \sum_{i=1}^{n}(|c_i| \cdot \Delta_i).$$

- In particular, this means that the sum, product, etc., of two intervals is now viewed as a twin interval.
- We can then use formulas for decision making under twin interval uncertainty to make decisions.

11. Proof: Main Ideas

- Reminder: $\overline{y}^{\,-} = \inf\{\overline{y}(S) : \pi_i(S) = [\underline{x}_i, \overline{x}_i]$ for all $i\}$, where $\overline{y}(S) \stackrel{\text{def}}{=} \sup\{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in S\}$.
- Here, $f(x_1, \ldots, x_n) = \widetilde{y} + \sum\limits_{i=1}^{n} c_i \cdot \Delta x_i$ and

$$\Delta x_i \in [-\Delta_i, \Delta_i].$$

- Without losing generality, we can assume that

$$\max_i (|c_i| \cdot \Delta_i) = |c_1| \cdot \Delta_1.$$

- In this case, the desired formula takes the form

$$\overline{y}^{\,-} = \widetilde{y} + |c_1| \cdot \Delta_1 - \sum_{i=2}^{n} |c_i| \cdot \Delta_i.$$

12. Proof: First Part

- Let us first prove, by contradiction, that
$$\overline{y}(S) \geq \widetilde{y}+|c_1|\cdot\Delta_1-\sum_{i=2}^{n}|c_i|\cdot\Delta_i \text{ for all } S \text{ s.t. } \pi_i(S)=[\underline{x}_i,\overline{x}_i].$$
- Indeed, let us assume that for some S with $\pi_i(S) = [\underline{x}_i,\overline{x}_i]$, we have $\overline{y}(S) < \widetilde{y} + |c_1|\cdot\Delta_1 - \sum_{i=2}^{n}|c_i|\cdot\Delta_i$.
- Since $\overline{y}(S) \stackrel{\text{def}}{=} \sup\{f(x_1,\ldots,x_n) : (x_1,\ldots,x_n) \in S\}$, this means that for all $x \in S$:
$$f(x_1,\ldots,x_n) \leq \overline{y}(S) < \widetilde{y} + |c_1|\cdot\Delta_1 - \sum_{i=2}^{n}|c_i|\cdot\Delta_i.$$
- By definition of $f(x_1,\ldots,x_n)$, this means that
$$\widetilde{y} + \sum_{i=1}^{n} c_i\cdot\Delta x_i < \widetilde{y} + |c_1|\cdot\Delta_1 - \sum_{i=2}^{n}|c_i|\cdot\Delta_i.$$

13. Proof: First Part (cont-d)

- So, $\sum\limits_{i=1}^{n} c_i \cdot \Delta x_i < |c_1| \cdot \Delta_1 - \sum\limits_{i=2}^{n} |c_i| \cdot \Delta_i$.
- Here, $\sum\limits_{i=2}^{n} c_i \cdot \Delta x_i \ge -\sum\limits_{i=2}^{n} |c_i| \cdot \Delta_i$, hence

$$-\sum_{i=2}^{n} c_i \cdot \Delta x_i \le \sum_{i=2}^{n} |c_i| \cdot \Delta_i.$$

- By adding these two inequalities, we conclude that

$$c_1 \cdot \Delta x_1 < |c_1| \cdot \Delta_1.$$

- Since $\pi_1(S) = [\underline{x}_1, \overline{x}_1] = [\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1]$, there is $x \in S$ for which $\Delta x_1 = \Delta_1 \cdot \operatorname{sign}(c_1)$.
- But for this x, we have $c_1 \cdot \Delta_1 = |c_1| \cdot \Delta_1$, a contradiction.
- So, the desired inequality is proven.

14. Proof: Second Part

- To complete the proofs, we need to show that for some set S with $\pi_i(S)=[\underline{x}_i,\overline{x}_i]$, we have

$$\overline{y}(S)\le \widetilde{y}+|c_1|\cdot\Delta_1-\sum_{i=2}^{n}|c_i|\cdot\Delta_i.$$

- As such S, let us take $S=\bigcup\limits_{i=1}^{n} S_i$, where

$$S_i=\{x : |\Delta x_1|\le\Delta_1\ \&\ \Delta x_j=-\Delta_j\cdot\text{sign}(c_j)\text{ for all } j\ne i\}.$$

- Here, $\pi_i(S_i)=[\underline{x}_i,\overline{x}_i]$, so $\pi_i(S)=[\underline{x}_i,\overline{x}_i]$ for all i.
- For every i and for every $x\in S_i$, we have

$$f(x_1,\ldots,x_n)=\widetilde{y}+\sum_{j=1}^{n}c_j\cdot\Delta x_j=c_i\cdot\Delta x_i-\sum_{j\ne i}|c_j|\cdot\Delta_j.$$

15. Proof: Second Part (cont-d)

- *Reminder:* for every $x \in S_i$, we have

$$f(x_1,\dots,x_n) = \widetilde{y} + \sum_{j=1}^{n} c_j \cdot \Delta x_j = c_i \cdot \Delta x_i - \sum_{j\ne i} |c_j| \cdot \Delta_j.$$

- Since $c_i \cdot \Delta x_i \le |c_i| \cdot \Delta_i$ we get

$$f(x_1,\dots,x_n) \le |c_i|\cdot\Delta_i - \sum_{j\ne i} |c_j|\cdot\Delta_j = 2|c_i|\cdot\Delta_i - \sum_{j=1}^{n} |c_j|\cdot\Delta_j.$$

- We know that $|c_i| \cdot \Delta_i \le |c_1| \cdot \Delta_1$ – this is how we selected x_1; thus,

$$f(x_1,\dots,x_n) \le 2|c_1|\cdot\Delta_1 - \sum_{j=1}^{n} |c_j|\cdot\Delta_j = |c_1|\cdot\Delta_1 - \sum_{j\ne i} |c_j|\cdot\Delta_j.$$

- The result is proven.
- For $\underline{y}^+$, the proof is similar.

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