Finding Zeros for Systems of Analytic Functions

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Introduction

In collaboration with

Dr. Marcelo Ciappina, Max Planck Institute for Quantum Optics
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Goal:
Find all zeros to systems of (two) analytic functions.

The Algorithm

The method we will use to find the zeros can be divided into two parts.

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1. Calculate the number of zeros in the domain.
2. Perform an exhaustive random search for the zeros and validate them.

The logarithmic integral (the argument principle)

Let $D \subset \mathbb{C}$ be a bounded domain with piecewise smooth boundary, ∂D. For an analytic function $f : \overline{D} \to \mathbb{C}$, with no zeros on the boundary, the number of zeros in D is given by

$$\frac{1}{2\pi} \int_{\partial D} \frac{f'(z)}{f(z)} dz.$$

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For multiple complex variables we need a more general formula.

Multidimensional logarithmic integral

Let $D \subset \mathbb{C}^n$ be a bounded domain with piecewise smooth boundary, ∂D. For an analytic function $f : \overline{D} \to \mathbb{C}^n$, with no zeros on the boundary, the number of zeros in D is given by

$$\frac{(n-1)!}{(2\pi i)^n} \int_{\partial D} \frac{1}{|f|^{2n}} \sum_{j=1}^{n} (-1)^{j-1} \overline{f}_j d\overline{f}_{[j]} \wedge df.$$

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To be able to evaluate it on a computer we need to rewrite it.

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$$\frac{(-1)^{k-1}}{4\pi^2}\int_{\partial D_i}\frac{\det J_f \overline{\det J_{[k]}}}{|f|^4}d\overline{z_{[k]}}dz_1dz_2$$

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Now all parts of the integral can be evaluated on a computer.

Calculate the Integral

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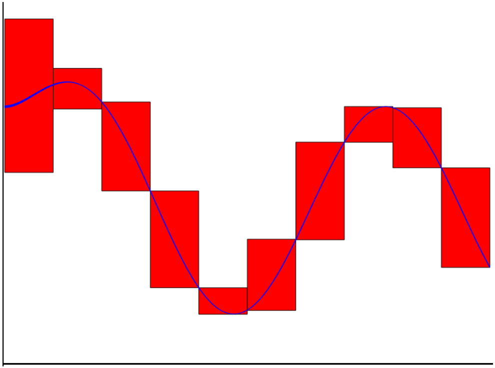
We will make it simple

$$base \cdot height$$

$$(volume \cdot height)$$



Calculate the Integral

A simple example

Calculate the number of zeros of $f(z_1, z_2) = (z_1, z_2)$ in $D = ([-1, 1] + i[-1, 1]) \times ([-1, 1] + i[-1, 1])$.

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Calculate the first integral.

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$$ ./integrate -1 -1 -1 1 -1 1 -1 1 0 0.125
Time used: 15.948 msec
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so the function must have exactly 1 zero in the domain.

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The integral can be very hard to evaluate!

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This will be done using a combination of Newton's method and the validated version of Newton's method.

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In practice we will perform Newton's method to several points, and then check them all.

Examples

Function

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\begin{aligned}
f(z_1, z_2) =& (\sin(z_1) + z_1^2 + e^{z_2} - \cos(2z_2), \cos(z_1) + z_2^3 + e^{2z_2} - 2) \\
D =& ([-1, 1] + i[-1, 1]) \times ([-1, 1] + i[-1, 1])
\end{aligned}
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Integral value = ([ 2.668302, 4.593785],[ -0.772578, 0.776699])
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- It then finds 4 unique zeros.

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where

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A(z) =& A_0 \cos(\omega_0 z + \phi) + A_1 \cos(\omega_1 z + \phi) + A_2 \cos(\omega_2 z + \phi), \\
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$$k_s(z_1, z_2) = -\frac{A'(z_2) - A'(z_1)}{z_2 - z_1}$$

and

$$D = ([0, 450] + i[0, 450]) \times ([0, 450] + i[0, 450]).$$

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- 240000 points were generated.

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- 240000 points were generated.
- Newtons interval method were used on 120000 respectively 240000 of these.

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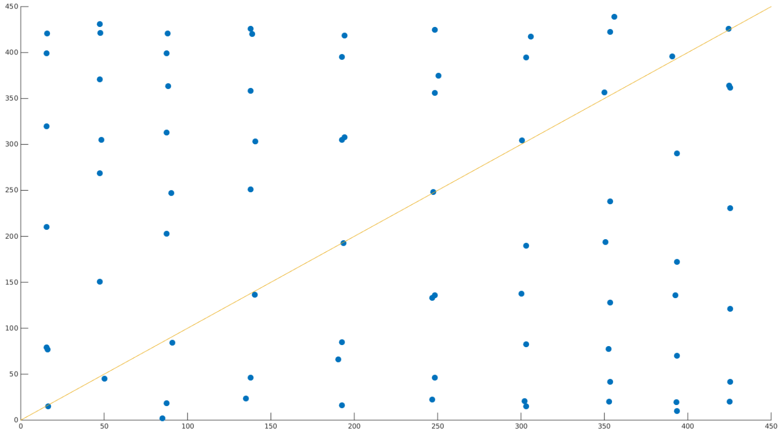
- 240000 points were generated.
- Newtons interval method were used on 120000 respectively 240000 of these.
- In the first case 79 zeros were isolated, in the second 80.



0
50
100
150
200
250
300
350
400
450

Future development

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Handling of multiple zeros and clusters of zeros

- Develop a method for locating such areas.
- Use the logarithmic integral for proving the existence of them.