A parametric Kantorovich theorem with application to tolerance synthesis [GCC15]

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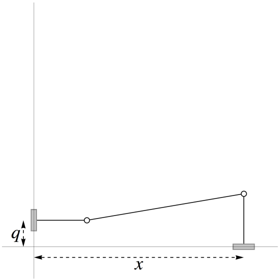
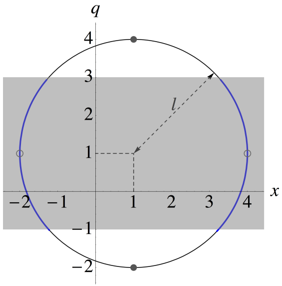
Motivation





$$\begin{aligned}(x-1)^2+(q-1)^2&=3^2\\ -3\le x\le 5\wedge -1\le q&\le 3\end{aligned}\tag{1}$$

Parallel robot

- Kinematic model: system of equations $f(x,q,p)=0$ with
  - $x\in\mathbb{R}^n$ the pose
  - $q\in\mathbb{R}^m$ the command (often $m=n$)
  - $p\in\mathbb{R}^q$ uncertainties on system parameters
  - $f(x,q,p)\in\mathbb{R}^n$ (as many equations as pose coordinates)
- Nominal workspace: $\mathcal{W}:=\{(x,q)\in\mathbb{R}^n\times\mathbb{R}^m : f(x,q,0)=0, g(x,q)\le 0\}$

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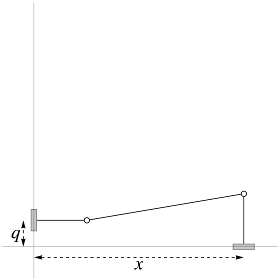

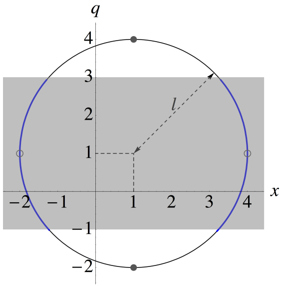

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Direct kinematic problem

- q and p fixed, compute x_p such that $f(x_p, q, p) = 0$
- **Square system of equations**
  - $p = 0 \Longrightarrow x_0$ is a nominal solution
  - $p \neq 0 \Longrightarrow x_p$ is a perturbed solution

Tolerance analysis

- Given $\Delta \geq \|p\|$
- Find $\overline{\epsilon}$ maximal distance between x_0 and x_p in the workspace

$\longleftarrow$ Typical result

- $\epsilon(p)$ in addition to $\overline{\epsilon}$
- Too large p make no sense

$\Leftrightarrow$ Is Δ small enough?

$\Rightarrow$ Need to compute a domain for p too

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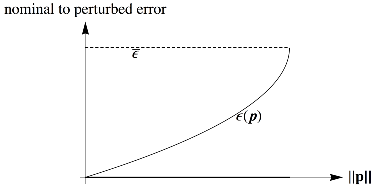

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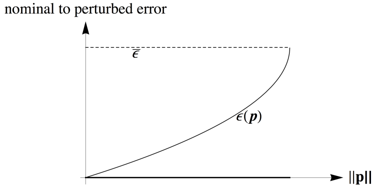

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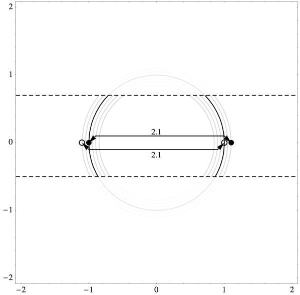

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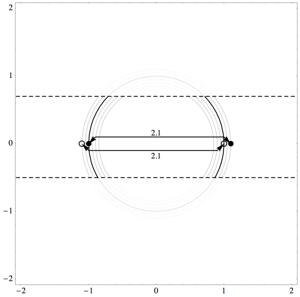

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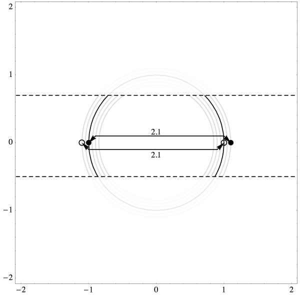

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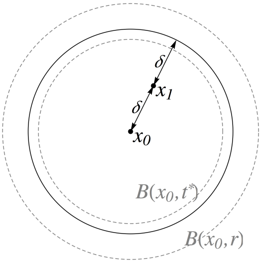
Kantorovich Theorem (simplified version)

Solve $f(x) = 0$

- Given x_0 compute the first Newton step
  $x_1 = x_0 + Df(x_0)^{-1} f(x_0)$
- Kantorovich constants χ, δ and λ
  - $\chi \geq \|Df(x_0)^{-1}\|$
  - $\delta \geq \|Df(x_0)^{-1} f(x_0)\| = \|x_1 - x_0\|$
  - λ Lipschitz constant for Df inside $B(x_0, r^+)$ with $r \geq 2\delta$



Kantorovich theorem

- $2\chi\delta\lambda \leq 1$ implies
  - $\exists x \in \overline{B}(x_0, t^*)$ with $t^*(\chi, \delta, \lambda) = \frac{1-\sqrt{1-2\delta\lambda\chi}}{\lambda\chi} \in [\delta, 2\delta]$
  - The solution is unique inside $\overline{B}(x_0, 2\delta)$

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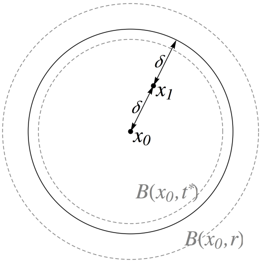

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Application to sensitivity analysis: Crude upper bound

Classical trick

- Bound on $\|x_0 - x_p\|$ is sought

$\Rightarrow$ Start Newton iterate at a nominal solution for solving the perturbed problem

- x_0 satisfies $f(x_0, q, 0) = 0$
- Kantorovich constants are computed for solving $f(x, q, p) = 0$

$\Rightarrow$ $t^* \equiv$ upper bound nominal/perturbed solution

Evaluation Kantorovich constants using global optimization

- Don't need to compute every x_0 to evaluate Kantorovich constants !

$\Rightarrow$ **Worst case constants for all (x, q) and p hold for every nominal solution**

$$\chi \geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1}\| \qquad \delta_0 \geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1} f(x,q,p)\|$$

- Lipschitz constant: worst case inside $\overline{B}(x_0, (2\delta_0)^+)$ for all $(x_0, q) \in \mathcal{W}$

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Application to sensitivity analysis: Sharp upper bound

Dependence of the first Newton step wrt $\|p\|$

- $f(x_0, q, p) = f(x_0, q, 0) + D_p f(x_0, q, 0)p + z = D_p f(x_0, q, 0)p + z$
- $\|z\| \leq \frac{1}{2}\mu\|p\|^2$ where μ is a Lipschitz constant for $D_p f$
  (for the whole workspace and all p such that $\|p\| \leq \Delta$)

$\Rightarrow$ $\|D_x f(x, q, p)^{-1} f(x_0, q, p)\| \leq \|D_x f(x, q, p)^{-1} D_p f(x_0, q, 0)\|\|p\| + \frac{1}{2}\chi\mu\|p\|^2$

- Worst case in workspace:

$$\delta_1(p) = \gamma\|p\| + \frac{1}{2}\chi\mu\|p\|^2 \quad \text{with}$$

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- Kantorovich applies if $2\chi\lambda\,\delta(p) \leq 1$ with $\delta(p) := \min\{\delta_0, \delta_1(p)\}$

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Parametric Kantorovich theorem

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$$\forall(x_0,q)\in\mathcal{W}, \forall p\in\overline{B}_\Delta, \forall x',x''\in\overline{B}(x_0,(2\delta_0)^+), \|D_x f(x',q,p)-D_x f(x'',q,p)\|\leq\lambda\|x'-x''\|$$

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$\delta_1(p)=\gamma\|p\|+\frac{1}{2}\chi\mu\|p\|^2$, $\delta(p)=\min\{\delta_0,\delta_1(p)\}$, $\mathcal{P}=\{p\in\mathbb{R}^q : \|p\|\leq\Delta\ ,\ 2\chi\lambda\delta(p)\leq 1\}$.

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The PRRP robot

Description

- $f(x,q,p) = (x-a-p_1)^2 + (q-b-p_2)^2 - (l+p_3)^2$, $a=1$, $b=1$ and $l=3$
- $\mathcal{W} = \{(x,q) \in \mathbb{R}^2 : f(x,q,p) = 0 \wedge x \in [-3,5] \wedge q \in [-1,3]\}$
- A priori maximal perturbation: $\Delta = 0.3$

Parametric Kantorovich constants definition

$$\chi \geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \frac{1}{2\,|x-a-p_1|}$$

$$\delta \geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \frac{|\,(x-a-p_1)^2 + (q-b-p_2)^2 - (l+p_3)^2\,|}{2\,|x-a-p_1|}$$

$$\gamma \geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \frac{|x-a-p_1| + |q-b-p_2| + |l+p_3|}{|x-a-p_1|}.$$

f quadratic wrt x and $p \Rightarrow$ Lipschitz constants $\lambda = 2$ and $\mu = 2$

The PRRP robot

Description

- $f(x,q,p) = (x-a-p_1)^2 + (q-b-p_2)^2 - (l+p_3)^2$, $a=1$, $b=1$ and $l=3$
- $\mathcal{W} = \{(x,q) \in \mathbb{R}^2 : f(x,q,p) = 0 \wedge x \in [-3,5] \wedge q \in [-1,3]\}$
- A priori maximal perturbation: $\Delta = 0.3$

Parametric Kantorovich constants definition

$$
\begin{aligned}
\chi &\geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \frac{1}{2\,|x-a-p_1|} \\
\delta &\geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \frac{|\,(x-a-p_1)^2 + (q-b-p_2)^2 - (l+p_3)^2\,|}{2\,|x-a-p_1|} \\
\gamma &\geq \max_{\substack{(x,q)\in\mathcal{W} \\ \|p\|\leq\Delta}} \frac{|x-a-p_1| + |q-b-p_2| + |l+p_3|}{|x-a-p_1|}\,.
\end{aligned}
$$

f quadratic wrt x and $p \Rightarrow$ Lipschitz constants $\lambda = 2$ and $\mu = 2$

The PRRP robot

Parametric Kantorovich constants computation

- Nonlinear non-smooth problems
- IBEX[a] [CJ09, TANC11, ATNC14]: branch and bound + numerical constraint programming + linearization: $\chi = 0.26$, $\delta_0 = 1.1$ and $\gamma = 3.90$ (less than 0.01s)

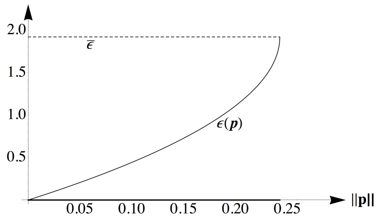
[a] Available at `http://www.ibex-lib.org/`

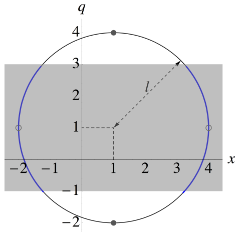
Bounds

- $\delta_1(p) = 3.9p + 0.26p^2$
- $\mathcal{P} = \{p \in \mathbb{R}^3 : \|p\| \leq 0.24\}$
- $\overline{\epsilon} = 1.93$
- $\epsilon(p) = 1.93(1 - \sqrt{1 - 1.04\delta(p)})$

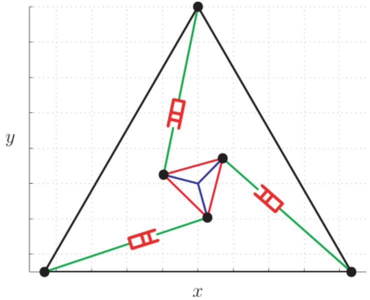
  $\epsilon(p) \approx \gamma\|p\|$ for $\|p\| \ll 1$

  (correct asymptotic is 3.82 $\rightarrow$ overestimated)





The 3R<u>P</u>R robot



- $n = m = 3$
- $+$ Parameters grouped into two classes
- $+$ Orientation and position errors
- Constants computed in less than a minute
- Perturbations allowed ≤ 0.002
- $\overline{\epsilon} = 0.03$

$$(p_1 - x_1)^2 + (p_2 - x_2)^2 - (p_3 + q_1)^2$$
$$\Big(L + p_4 - x_1 - l\sin(\frac{\pi}{6} + x_3)\Big)^2 + \Big(p_5 - x_2 + l\cos(\frac{\pi}{6} + x_3)\Big)^2 - (p_6 + q_2)^2$$
$$\Big(\frac{L}{2} + p_7 - x_1 - l\cos(x_3)\Big)^2 + \Big(\frac{L\sqrt{3}}{2} + p_8 - x_2 - l\sin(x_3)\Big)^2 - (p_9 + q_3)^2,$$
$$\text{with } L = 1 \text{ and } l = 0.5$$

The 3RPR robot: Minibex code for χ

```
constants

dmax1=0.01;
dmax2=0.01;
Pi in [3.14159265358979,3.14159265358980];

Variables

x1 in [-0.55,0.05];
x2 in [1.2,1.8];
x3 in [0.,0.];
q1 in [0.,100.];
q2 in [0.,100.];
q3 in [0.,100.];
d1 in [-dmax1,dmax1];
d2 in [-dmax1,dmax1];
d4 in [-dmax1,dmax1];
d5 in [-dmax1,dmax1];
d7 in [-dmax1,dmax1];
d8 in [-dmax1,dmax1];
d3 in [-dmax2,dmax2];
d6 in [-dmax2,dmax2];
d9 in [-dmax2,dmax2];

Minimize

-( max(max(2*abs(sqrt(3)*d1 - d2 - 2*d2*d7 + 2*d1*d8 - sqrt(3)*x1 + 2*d2*x1 - 2*d8*x1 + x2 - 2*d1*x2 + 2*d7*x2 + (d2 -
x2)*cos(x3) + (-d1 + x1)*sin(x3)) + abs(2*(d1 - x1)*(2*d5 - 2*x2 + cos(Pi/6. + x3)) - 2*(d2 - x2)*(2 + 2*d4 - 2*x1 -
sin(Pi/6. + x3))) + abs(-((1 + 2*d7 - 2*x1 - cos(x3))*(2*d5 - 2*x2 + cos(Pi/6. + x3))) + (sqrt(3) + 2*d8 - 2*x2 -
sin(x3))*(2 + 2*d4 - 2*x1 - sin(Pi/6. + x3))),abs((d1 - x1)*((sqrt(3) + 2*d8 - 2*x2)*cos(x3) + (-1 - 2*d7 +
2*x1)*sin(x3))) + 2*abs((d1 - x1)*((-1 - d4 + x1)*cos(Pi/6. + x3) + (-d5 + x2)*sin(Pi/6. + x3))) + abs((((sqrt(3) +
2*d8 - 2*x2)*cos(x3) + (-1 - 2*d7 + 2*x1)*sin(x3))*(-2 - 2*d4 + 2*x1 + sin(Pi/6. + x3)))/2. - (1 + 2*d7 - 2*x1 -
cos(x3))*((-1 - d4 + x1)*cos(Pi/6. + x3) + (-d5 + x2)*sin(Pi/6. + x3)))),abs((d2 - x2)*(-((sqrt(3) + 2*d8 -
2*x2)*cos(x3)) + (1 + 2*d7 - 2*x1)*sin(x3))) + 2*abs((d2 - x2)*((-1 - d4 + x1)*cos(Pi/6. + x3) + (-d5 + x2)*sin(Pi/6.
+ x3))) + abs(-((2*d5 - 2*x2 + cos(Pi/6. + x3))*(-((sqrt(3) + 2*d8 - 2*x2)*cos(x3)) + (1 + 2*d7 - 2*x1)*sin(x3)))/2. +
(sqrt(3) + 2*d8 - 2*x2 - sin(x3))*((-1 - d4 + x1)*cos(Pi/6. + x3) + (-d5 + x2)*sin(Pi/6. + x3))))/abs((-
(cos(x3)*(sqrt(3)/2. + d8 - x2 - sin(x3)/2.)) + (0.5 + d7 - x1 - cos(x3)/2.)*sin(x3))*(4*(d1 - x1)*(d5 - x2 + cos(Pi/
6. + x3)/2.) - 4*(d2 - x2)*(1 + d4 - x1 - sin(Pi/6. + x3)/2.)) - (2*sqrt(3)*d1 - 2*d2 - 4*d2*d7 + 4*d1*d8 -
2*sqrt(3)*x1 + 4*d2*x1 - 4*d8*x1 + 2*x2 - 4*d1*x2 + 4*d7*x2 + 2*d2*cos(x3) - 2*x2*cos(x3) - 2*d1*sin(x3) +
2*x1*sin(x3))*(-(cos(Pi/6. + x3)*(1 + d4 - x1 - sin(Pi/6. + x3)/2.)) - (d5 - x2 + cos(Pi/6. + x3)/2.)*sin(Pi/6. +
x3))) )

Constraints

-q1^2 + x1^2 + x2^2=0;
-q2^2 + (-x2 + cos(Pi/6. + x3)/2.)^2 + (1 - x1 - sin(Pi/6. + x3)/2.)^2=0;
-q3^2 + (0.5 - x1 - cos(x3)/2.)^2 + (sqrt(3)/2. - x2 - sin(x3)/2.)^2=0;

end
```

Conclusion

Strengths

- Parametric Kantorovich theorem
  - Solve 5 global optimization problems
  - $\Rightarrow$ Parameter domain for which perturbed solution remains uniquely defined
  - $\Rightarrow$ Upper bound on the nominal to perturbed solution distance
- Applies not only to one solution, but to a manifold of solutions

Weaknesses

- Timing increases with manifold dimension
  - ! Formal inverse is used
  - Not applicable for $n \geq 4$ (robots can have $n = 6$)
- Parameter domain decreases with manifold dimension
  - Up to now, 0.002 meters for 1 meter robot is ok

Future work

- Use interval matrix inverse (e.g., $M(x)A = I$ instead of $M(x)^{-1}$ in optimization problems)
- Test on robots with more dofs

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