Computing Attracting Ellipsoids for Nonlinear Systems using an Interval Lyapunov Equation

SWIM'2016

Léopold Houdin, Alexandre Goldsztejn, Gilles Chabert, Frédéric Boyer

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Outline

Context

We consider a continuous-time dynamical system :

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n,$$

and an *exponentially stable* point $x^\star$.

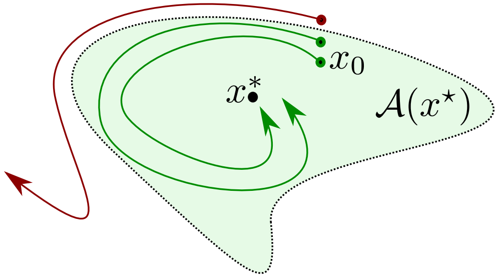
Context

Exponential Stability

All the trajectories starting inside a neighborhood $\mathcal{A}(x^\star)$ of $x^\star$ converge to $x^\star$ faster than an exponential decay :

$$\forall x_0 \in \mathcal{A}(x^\star),\ \exists t_0,\ \forall t \geq t_0, \quad \|x(t) - x^*\| \leq \alpha \|x_0 - x^*\| e^{-\beta t}.$$

for some nonnegative constants α, β.



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➡ These eigenvalues also characterize the attraction strength (β).

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$f(x)$	$\mathrm{D}f(0)$	$\mathcal{A}(0)$
$c^3x^3 - cx$	$-c$	$(-\frac{1}{c}, \frac{1}{c})$
$\frac{x^3}{c} - cx$	$-c$	$(-c, c)$

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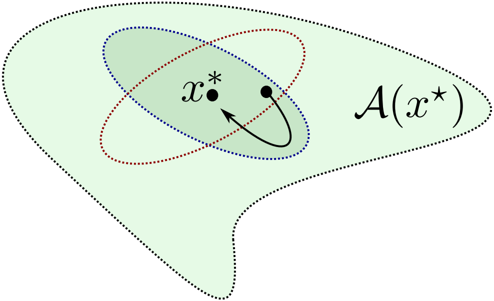
Overall Objective

Build a subset of $\mathcal{A}(x^\star)$ in a **fast** (polynomial time) and **guaranteed** way.



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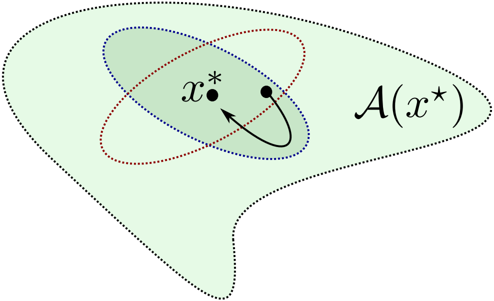

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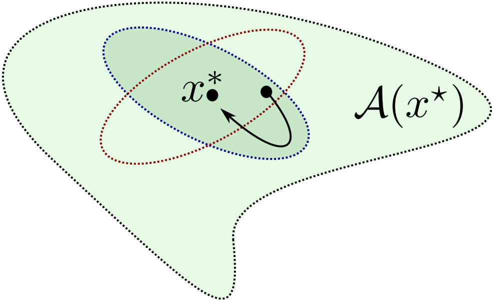


The subset must also be a **positive invariant**.

Our approach is based on **Lyapunov theory**.

Outline

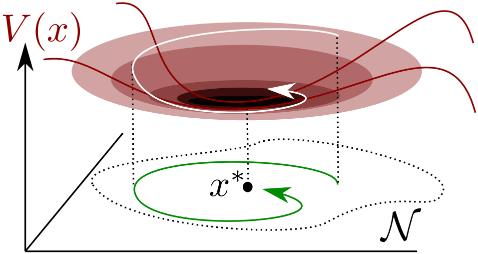
Lyapunov Functions

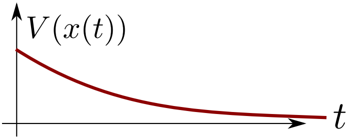
A Lyapunov function is locally an energy-like function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(x^\star) = 0$$

and there exists an open neighborhood $\mathcal{N}$ of x^* such that

$$\forall x \in \mathcal{N} \backslash \{x^\star\} \quad \begin{cases} V(x) > 0 \\ \dot{V}(x) < 0 \end{cases} \quad \text{with} \quad \dot{V}(x) := \tfrac{d}{dt} V(x(t)).$$





Lyapunov Functions

There always exists [1] a Lyapunov function of the form

$$V(x) = (x - x^\star)^T P (x - x^\star)$$

where P is a SPD (*symmetric positive definite*) matrix.

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In short,

$$V(x) = \|x - x^\star\|_P.$$

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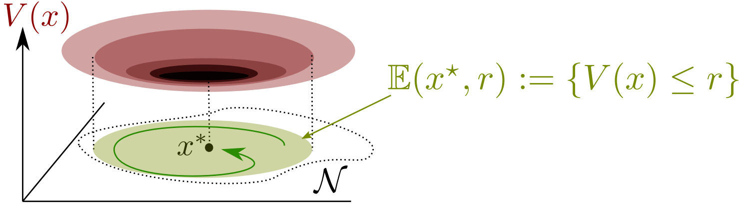
Lyapunov Functions

Then, all the level sets of V **inside** $\mathcal{N}$ are attracting, and even more, positive invariants.



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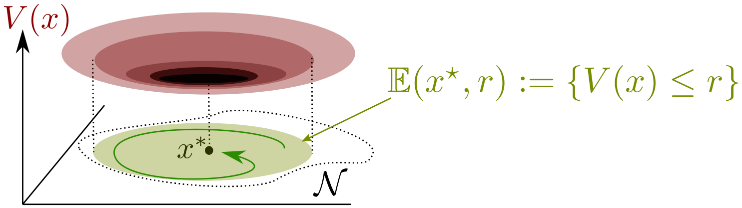

Problem : We know V but we don't know $\mathcal{N}$.

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Note :

$$\dot{V}(x) = \frac{d}{dt} V(x(t)) = \nabla V(x)^T f(x) = 2(x - x^\star)^T P f(x)$$

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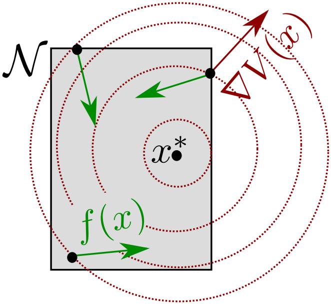

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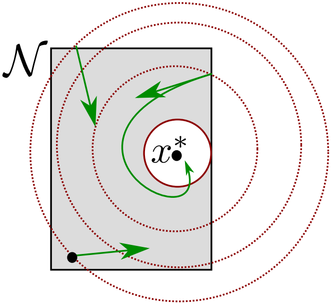

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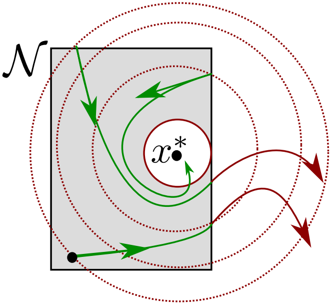

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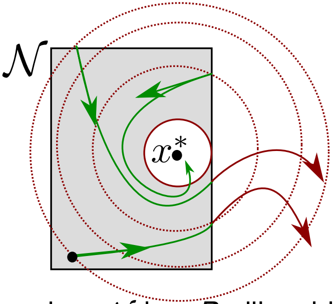

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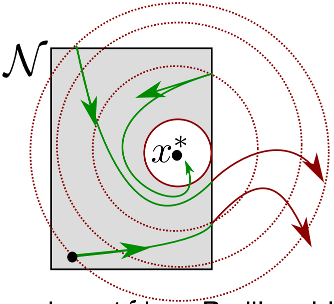

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➡ difficult with interval methods.

Concrete Objective

Interval approach

Set $\mathcal{N} := [x]$ (*an arbitrary box around* $x^\star$) and prove :

$$\forall x \in [x],\ x \neq x^\star, \quad \dot{V}(x) = (x - x^\star)^T P f(x) < 0.$$

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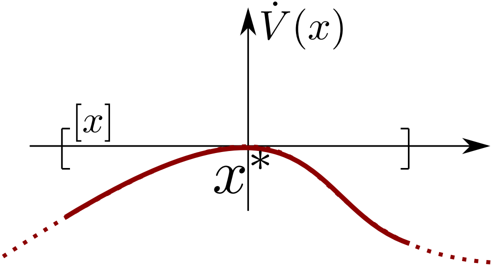
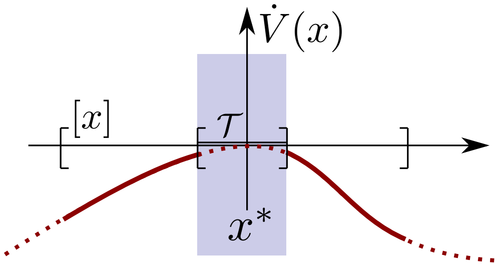
Problem : naively applying interval arithmetic does not work since $\dot{V}(x^\star) = 0$.



Concrete Objective

Ratschan & She proposed to “remove” a small region $\mathcal{T}$ (called *target*) around $x^\star$.



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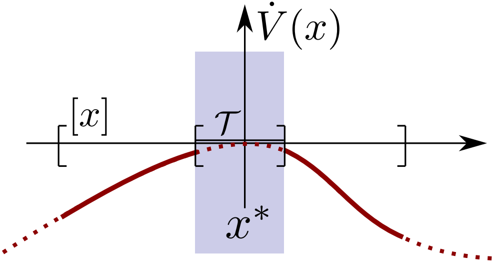

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all trajectories in a level set inside $[x]$ reach the target.

Problem : convergence to $x^\star$ is not proven.

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Delanoue, Jaulin and Cottenceau prove

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- Resorts to second-order derivatives.

Outline

Overview

Contribution.

We propose two algorithms :

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- The second uses 2nd order derivative but always gives a solution (under mild conditions)
- Both work directly with ellipsoids (inputs and outputs are radii)

Outline

Testing a Neighborhood

First, we know from the fundamental theorem of analysis that, for all $x \in \mathbb{R}^n$ there exists a matrix $S(x)$ such that

$$f(x) = f(x^\star) + S(x)(x - x^\star) = S(x)(x - x^\star)$$

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Proposition ("Interval Lyapuonv Equation")

$\forall x \in [x]$, define $Q(x)$ as follows :

$$Q(x) := S(x)^T P + PS(x).$$

If $Q([x])$ contains only ND matrices, then $\mathcal{N} := [x]$ is a valid neighborhood.

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$$\forall x \in [x], \quad \begin{aligned} \dot{V}(x) &= 2(x - x^\star)^T P f(x) \\ &= 2(x - x^\star)^T P S(x)(x - x^\star) \\ &= (x - x^\star)^T (S(x)^T P + P S(x))(x - x^\star) \end{aligned}$$

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and since $Q(x)$ is ND by hypothesis, $x \neq x^\star \Longrightarrow \dot{V}(x) < 0$.

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➡ And take into account the uncertainty on $x^{\star}$.

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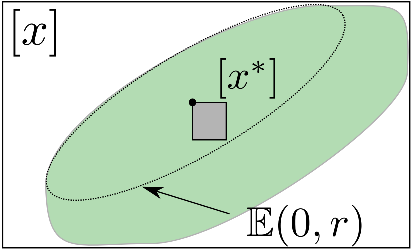
Building an Ellispoid

So we have first to find the largest value of r such that

$$[x] \supseteq [x^{\star}] + \square\left(\mathbb{E}(0, r)\right)$$

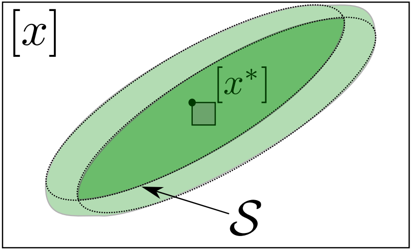


Building an Ellispoid

And then build the set

$$\mathcal{S} := \bigcap_{x \in [x^\star]} x + \square\left(\mathbb{E}(0, r)\right)$$

which is attracting.



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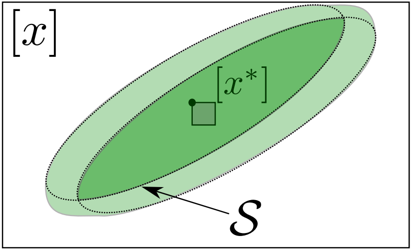

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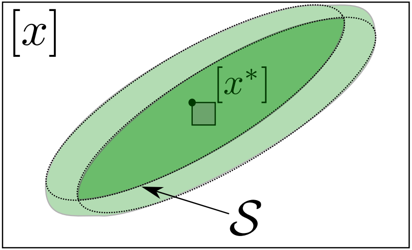

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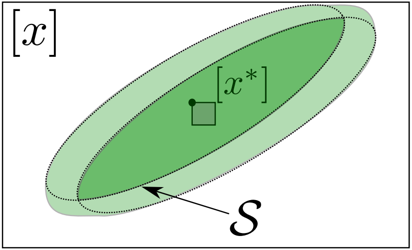

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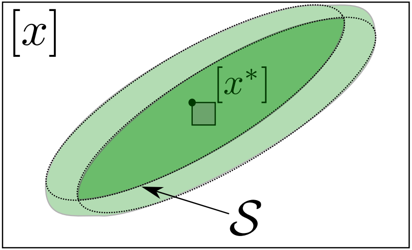

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Problem : $\mathcal{S}$ is not an ellipsoid (and difficult to compute).

Building an Ellispoid

Idea : we proceed in the other way round.

We start from a candidate ellipsoid

$$\mathbb{E}(\hat{x}, \hat{r})$$

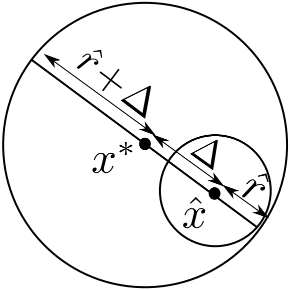
with $\hat{x} \in [x^\star]$ and build the box $[x]$ accordingly.

Building an Ellispoid

Define

$$[\Delta] := \left\| [x^\star] - \hat{x} \right\|_P .$$

Thanks to the triangular inequality, we have :



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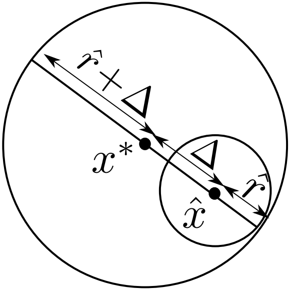

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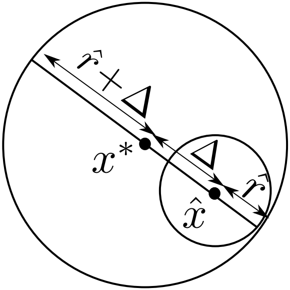
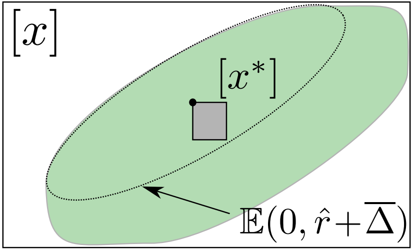


- $\mathbb{E}(\hat{x}, \hat{r}) \subseteq \mathbb{E}(x^\star, \hat{r} + \overline{\Delta})$.
- If $\overline{\Delta} \leq \hat{r}$ then $x^\star \in \mathbb{E}(\hat{x}, \hat{r})$.

Building an Ellispoid

Define

$$[x] := [x^\star] + \square\left(\mathbb{E}(0, \hat{r} + \overline{\Delta})\right)$$



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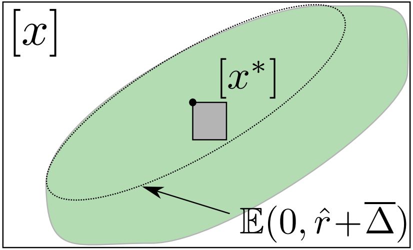

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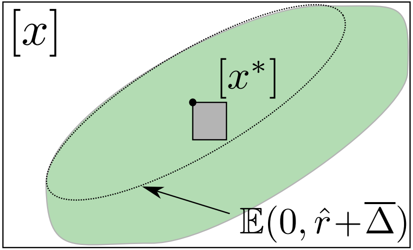

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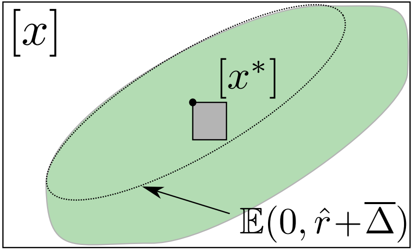

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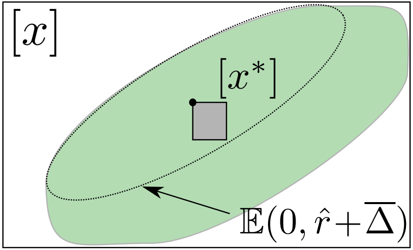

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using bounds L and L' on the P-norm Lipsichitz constants of S and S^T, on an initial arbirary box $[x]$.

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➡ We need an algorithm for rigorously bounding eigenvalues (e.g. interval variant of Gerschgorin's circles)

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- \+ Even if the fixpoint is approximately known
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- \- Invariance is lost in theory (but not in practice)

Thanks !

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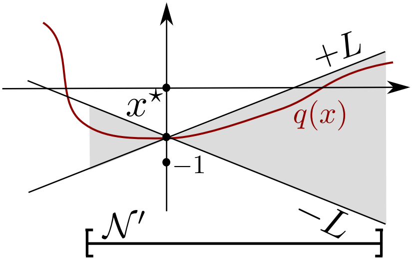
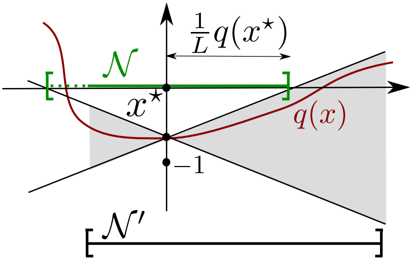
- a Lipschitz bound L on q over an (arbirary initial) set $\mathcal{N}'$:

$$\forall x, y \in \mathcal{N}' \quad \|q(x) - q(y)\| \leq L\|x - y\|.$$



Then

$$\left(x \in \mathcal{N}' \wedge \|x - x^\star\| < \frac{1}{L} q(x^\star)\right) \Longrightarrow q(x) < 0$$



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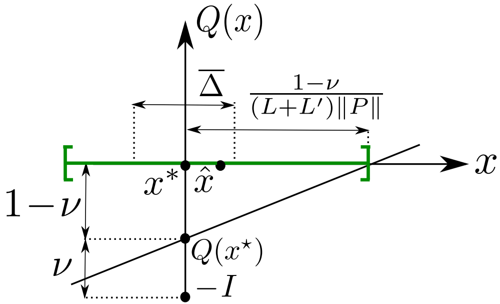

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