Interval tools for computing the topology of projected curves

Rémi Imbach, Guillaume Moroz and Marc Pouget

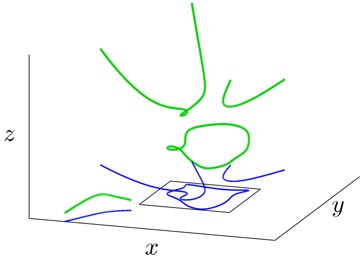
Projection and Apparent Contour

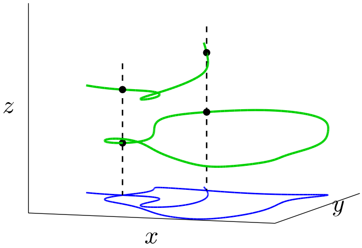
P, Q two polynomial maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) &= 0 \\ Q(x,y,z) &= 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\pi_{(x,y)}(\mathcal{C})$





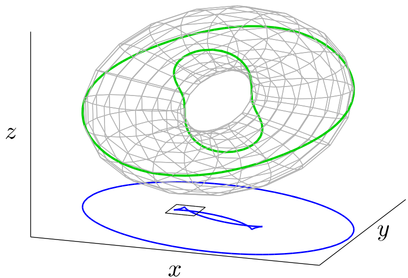
Projection and Apparent Contour

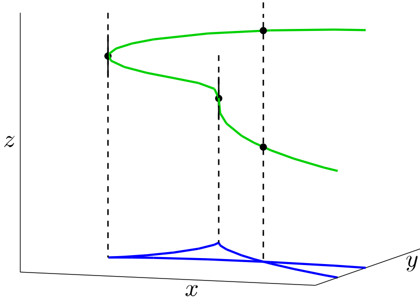
P, Q two polynomial maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) &= 0 \\ P_z(x,y,z) &= 0 \end{cases}, (x,y,z) \in \mathbb{R}^3, \qquad P_z = \frac{\partial P}{\partial z}$$

Apparent contour: $\pi_{(x,y)}(\mathcal{C})$



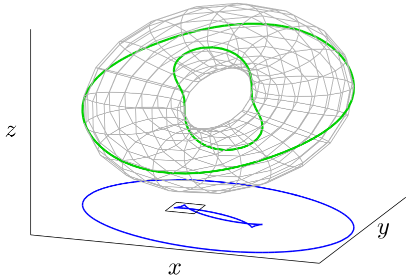


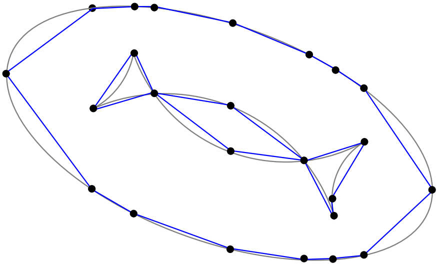
Computing the topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Goal: with numerical approaches, compute

- exact topology
- approximated geometry



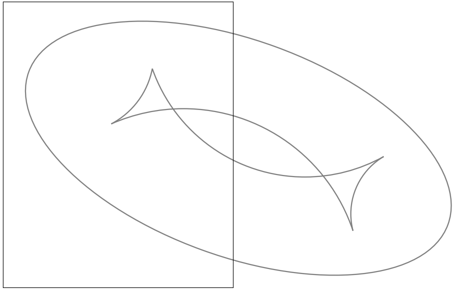


Computing the topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Goal: with numerical approaches, compute

- exact topology
- approximated geometry



A general framework

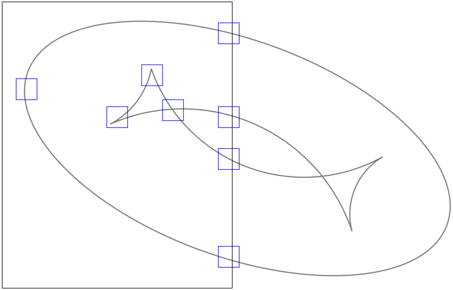
0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

Computing the topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Goal: with numerical approaches, compute

- exact topology
- approximated geometry



A general framework

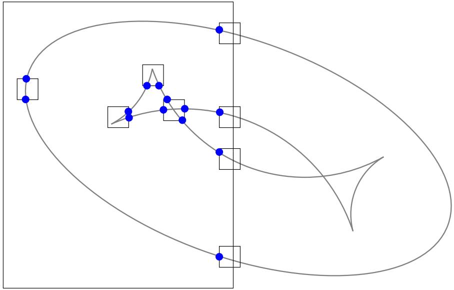
0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

Computing the topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Goal: with numerical approaches, compute

- exact topology
- approximated geometry



A general framework

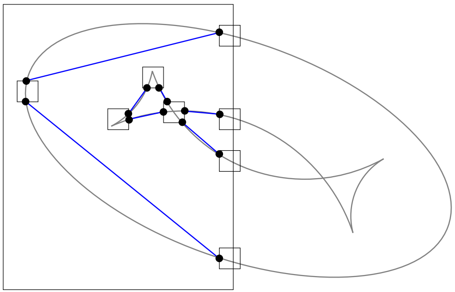
0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

Computing the topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Goal: with numerical approaches, compute

- exact topology
- approximated geometry



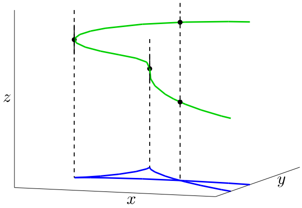
A general framework

0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

Computing the topology of a real plane curve $\mathcal{B}$

Characterization and isolation of nodes and cusps:

- Resultant approaches
- Geometric approach



❶ Isolate in boxes:

- boundary points
- x-critical points
- singularities

Certified numerical tools:

- 0-dim solver: branch and bound solver

Computing the topology of a real plane curve $\mathcal{B}$

Characterization and isolation of nodes and cusps:

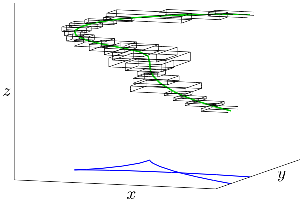
- Resultant approaches
- Geometric approach

Enclosing $\mathcal{C}$ in a sequence of boxes:

- Restrict the domain where singularities are sough
- Compute topology

Certified numerical tools:

- 0-dim solver: branch and bound solver
- 1-dim solver: certified path tracker



❶ Isolate in boxes:
- boundary points
- x-critical points
- singularities

❷ Compute topology around singularities

❸ Connect boxes

Isolating singularities

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$,

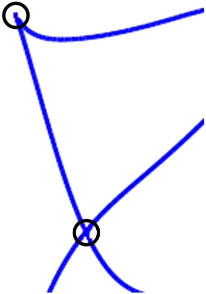
Singularities of $\mathcal{B}$ are the solutions of:

$$(\mathcal{S}) \begin{cases} r(x, y) = 0 \\ \frac{\partial r}{\partial x}(x, y) = 0 \\ \frac{\partial r}{\partial y}(x, y) = 0 \end{cases}$$

. . . that is over-determined

. . . that has solutions of multiplicity 2

symbolic approaches: Gröbner Basis, RUR



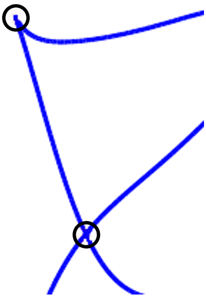
Isolating singularities of apparent contours

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the solutions of:

$$(\mathcal{S}) \begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{cases}$$

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials



symbolic approaches: Gröbner Basis, RUR

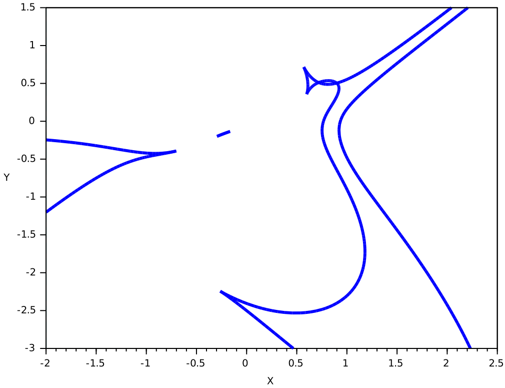
Example

P, degree 6, bit-size 8, 84 monomials

$$p = 158x^6 - 186x^5y + 205x^5z - 160x^4y^2 + 105x^4yz + 116x^4z^2 - 69x^3y^3 - 161x^3y^2z - 8x^3yz^2 + 107x^3z^3 + 144x^2y^4 - 193x^2y^3z + 130x^2y^2z^2 + x^2yz^3 + 47x^2z^4 + 165xy^5 - 220xy^4z - 21xy^3z^2 + 50xy^2z^3 - 130xyz^4 - 77xz^5 + 66y^6 - 55y^5z + 219y^4z^2 - 30y^3z^3 - 162y^2z^4 - 182yz^5 - 145z^6 + 105x^5 + 241x^4y - 177x^4z - 127x^3y^2 - 97x^3yz + 223x^3z^2 - 46x^2y^3 - 213x^2y^2z + 39x^2yz^2 + 191x^2z^3 + 65xy^4 - 105xy^3z - 248xy^2z^2 + 158xyz^3 - 183xz^4 + 48y^5 - 240y^4z + 235y^3z^2 + 194y^2z^3 - 45yz^4 + 159z^5 - 81x^4 - 230x^3y - 247x^3z - 38x^2y^2 + 106x^2yz + 184x^2z^2 + 49xy^3 - 197xy^2z - 182xyz^2 - 223xz^3 - 205y^4 - 225y^3z - 14y^2z^2 - 17yz^3 + 73z^4 - 234x^3 - 82x^2y + 179x^2z + 46xy^2 - 222xyz - 95xz^2 + 139y^3 + 168y^2z + 8yz^2 + 156z^3 + 159x^2 - 147xy - 22xz - 104y^2 + 181yz + 26z^2 - 90x + 250y + 19z + 19$$

Example

P, degree 6, bit-size 8, 84 monomials



Example

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials

$$Res_z(p, \frac{\partial p}{\partial z}) = 2537851751382193098537472618 5\,x^{30} - 1950289566984849821762662644 60\,x^{29}y + 66946066089386081392160455410 0\,x^{28}y^2 - 6313231163041522510562021480 00\,x^{27}y^3 - 10287045636804329902450223542 80\,x^{26}y^4 + 4597797015605117908624008082 0\,x^{25}y^5 + 35544695534063712937519877422 70\,x^{24}y^6 + 3711031010928440039666656612920\,x^{23}y^7 - 5634442800184514383998916600260\,x^{22}y^8 - 1165859185506938114470659584106 0\,x^{21}y^9 - 4387874939266072948066332459470\,x^{20}y^{10} + 1640884346103822842022302318023 0\,x^{19}y^{11} + 2370016579425177706230400977291 5\,x^{18}y^{12} + 4316324180997748865901800201620\,x^{17}y^{13} - 2492913730524765321908872849874 0\,x^{16}y^{14} - 3337290835102177803049211965481 0\,x^{15}y^{15} - 9633448028150975870147511674570\,x^{14}y^{16} + 2050015543179023515840337400119 0\,x^{13}y^{17} + 3166808906075930935068471645835 0\,x^{12}y^{18} + 1654427855021865261625001839852 0\,x^{11}y^{19} - 5014730522275651771719575652535\,x^{10}y^{20} - 1659011161494516371407397482332 0\,x^{9}y^{21} - 1354608334114918208346453586642 5\,x^{8}y^{22} - 4754759946941791724566012110130\,x^{7}y^{23} + 1997218841447789708607357568825\,x^{6}y^{24} + 3898998021968250822246999603270\,x^{5}y^{25} +$$

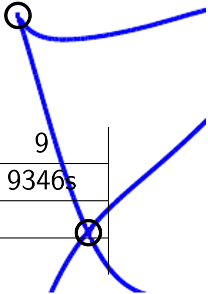
Isolating singularities of apparent contours

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the solutions of:

$$(\mathcal{S}) \begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{cases}$$

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials



symbolic approaches: Gröbner Basis, RUR

degree of P	6	7	8	9
$(\mathcal{S})$ with RSCube*	32s	254s	1898s	9346s

* F. Rouillier

Isolating singularities of apparent contours

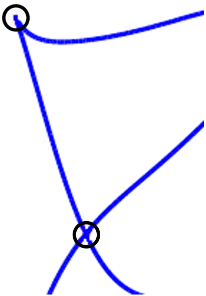
$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \quad \text{s.t. } s_{22}(x, y) \neq 0$$

. . . where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
s_{11}, s_{10},	degree 20,	bit-size 90,	231 monomials



[IMP16] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
A certified numerical algorithm for the topology of resultant and discriminant curves.
Journal of Symbolic Computation, 2016.

Example

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
s_{11}, s_{10},	degree 20,	bit-size 90,	231 monomials

$s11 = -14011784862700881253122 0 x^{20} - 6101531335933493541710 40 x^{19} y + 3951651892302173384407 0 x^{18} y^2 +$

$3342883727033466620154170 x^{17} y^3 + 2891274355142589403901890 x^{16} y^4 + 1127947297505275246498 40 x^{15} y^5 -$

$11340692490521298700125220 x^{14} y^6 - 11062911106388945165447000 x^{13} y^7 -$

$2946445042372334921153850 x^{12} y^8 + 1289064149306247575780837 0 x^{11} y^9 +$

$20482823881470123106468370 x^{10} y^{10} + 11024860229216130931420010 x^9 y^{11} -$

$1126962434297495978162860 x^8 y^{12} - 1288448532468574766443268 0 x^7 y^{13} -$

$9059725287074848327234580 x^6 y^{14} - 4941320817429025658253850 x^5 y^{15} + 2122391146412348698406760 x^4 y^{16} +$

$2384112136850068775369540 x^3 y^{17} + 2363347796938811648578260 x^2 y^{18} + 7359339415378012031667 20 x y^{19} +$

$293011939904302120871210 y^{20} + 693411445688541987909840 x^{19} + 4819667434476299196422270 x^{18} y -$

$8545316039998573100100 90 x^{17} y^2 - 4588903065796097271527060 x^{16} y^3 - 1245454007763298588704199 0 x^{15} y^4 -$

$19038809918580772113933260 x^{14} y^5 - 5255594134400598288192960 x^{13} y^6 + 11740052664047730440762 20 x^{12} y^7 +$

$39658021585466235582243720 x^{11} y^8 + 49141822061980186469013340 x^{10} y^9 +$

$5125145051120039185666669 0 x^9 y^{10} + 11669318785950916496923050 x^8 y^{11} -$

Isolating singularities of apparent contours

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \quad \text{s.t. } s_{22}(x, y) \neq 0$$

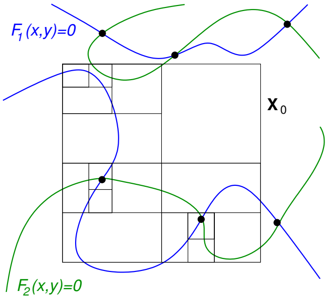
. . . where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
s_{11}, s_{10},	degree 20,	bit-size 90,	231 monomials

degree of P	6	7	8	9
$(\mathcal{S})$ with RSCube*	32s	254s	1898s	9346s
$(\mathcal{S}_2)$ with RSCube	15s	105s	620s	3 300s
$(\mathcal{S}_2)$ with Bertini	1005s	$\geq$ 3000s	$\geq$ 3000s	$\geq$ 3000s

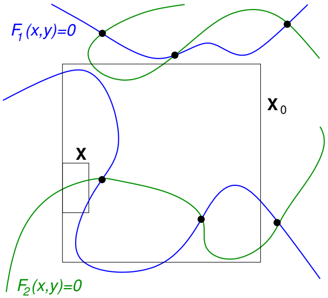
* F. Rouillier

A branch and bound solver for systems of large polynomials



[Kea96] R. Baker Kearfott.
Rigorous global search: continuous problems.
Nonconvex optimization and its applications. Kluwer Academic Publishers, Dordrecht, Boston, 1996.

A branch and bound solver for systems of large polynomials

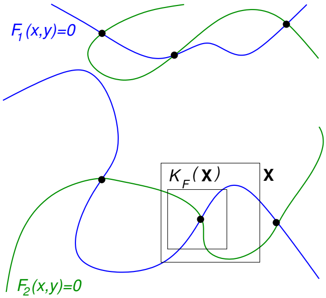


Interval extension $\square F$ of F:

$0 \notin \square F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

[Kea96] R. Baker Kearfott.
Rigorous global search: continuous problems.
Nonconvex optimization and its applications. Kluwer Academic Publishers, Dordrecht, Boston, 1996.

A branch and bound solver for systems of large polynomials



Interval extension $\square F$ of F:

$0 \notin \square F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

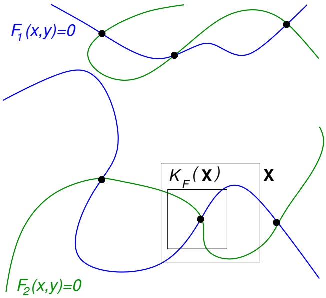
Interval newton operators $N_F(\mathbf{X})$:

$N_F(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \exists!$ solution in $\mathbf{X}$

- Interval Gauss-Seidel
- Krawczyk $K_F(\mathbf{X})$
- ...

[Kea96] R. Baker Kearfott.
Rigorous global search: continuous problems.
Nonconvex optimization and its applications. Kluwer Academic Publishers, Dordrecht, Boston, 1996.

A branch and bound solver for systems of large polynomials



Interval extension $\Box F$ of F:

$0 \notin \Box F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

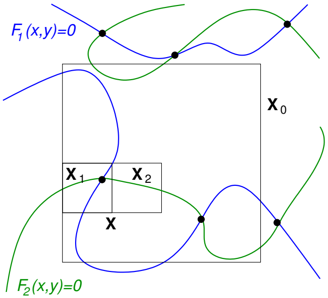
Interval newton operators $N_F(\mathbf{X})$:

$N_F(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \exists!$ solution in $\mathbf{X}$

- Interval Gauss-Seidel
- Krawczyk $K_F(\mathbf{X})$
- ...

[Kea96] R. Baker Kearfott.
Rigorous global search: continuous problems.
Nonconvex optimization and its applications. Kluwer Academic Publishers, Dordrecht, Boston, 1996.

A branch and bound solver for systems of large polynomials



Interval extension $\square F$ of F:

$0 \notin \square F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

Interval newton operators $N_F(\mathbf{X})$:

$N_F(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \exists!$ solution in $\mathbf{X}$

- Interval Gauss-Seidel
- Krawczyk $K_F(\mathbf{X})$
- ...

[Kea96] R. Baker Kearfott.
Rigorous global search: continuous problems.
Nonconvex optimization and its applications. Kluwer Academic Publishers, Dordrecht, Boston, 1996.

A branch and bound solver for systems of large polynomials

$s11 = -14011784862700881253122\,0 x^{20} - 610153133593349354171040\, x^{19}y +$

$395165189230217338440 70\, x^{18}y^2 + 334288372703346662015417 0\, x^{17}y^3 +$

$289127435514258940390189 0\, x^{16}y^4 + 112794729750527524649840\, x^{15}y^5 -$

$113406924905212987001252 20\, x^{14}y^6 - 110629111063889451654470 00\, x^{13}y^7 -$

$294644504237233492115385 0\, x^{12}y^8 + 128906414930624757578083 70\, x^{11}y^9 +$

$204828238814701231064683 70\, x^{10}y^{10} + 110248602292161309314200 10\, x^9y^{11} -$

$112696243429749597816286 0\, x^8y^{12} - 128844853246857476644326 80\, x^7y^{13} -$

$905972528707484832723458 0\, x^6y^{14} - 494132081742902565825385 0\, x^5y^{15} +$

$212239114641234869840676 0\, x^4y^{16} + 238411213685006877536954 0\, x^3y^{17} +$

$236334779693881164857826 0\, x^2y^{18} + 735933941537801203166720\, xy^{19} +$

$293011939904302120871210\, y^{20} + 693411445688541987909840\, x^{19} +$

$481966743447629919642227 0\, x^{18}y - 854531603999857310010090\, x^{17}y^2 -$

$458890306579609727152706 0\, x^{16}y^3 - 124545400776329858870419 90\, x^{15}y^4 -$

$190388099185807721139332 60\, x^{14}y^5 - 525559413440059828819296 0\, x^{13}y^6 +$

$117400526640477304407622 0\, x^{12}y^7 + 396580215854662355822437 20\, x^{11}y^8 +$

$491418220619801864690133 40\, x^{10}y^9 + 512514505112003918566666 90\, x^9y^{10} +$

$116693187859509164969230 50\, x^8y^{11} - 283014955365067922269880 80\, x^7y^{12} -$

Interval extension $\square F$ of F:

$0 \notin \square F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

Interval newton operators $N_F(\mathbf{X})$:

$N_F(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \exists!$ solution in $\mathbf{X}$

- Interval Gauss-Seidel
- Krawczyk $K_F(\mathbf{X})$
- ...

Main Issues:

Evaluating F:

- quickly
- sharply

Adapting arithmetic precision

Evaluating large multivariate polynomials

Quickly: multivariate Horner scheme

Sharply: $\Box F$ can be implemented by:

${}^0F$: Horner form,

${}^1F$: centered eval. at order 1,

${}^2F$: centered eval. at order 2,

[Neu90] A. Neumaier.
Interval methods for systems of equations.
Cambridge University Press, 1990.

Evaluating large multivariate polynomials

Quickly: multivariate Horner scheme

Sharply: $\square F$ can be implemented by:
^{0}F: Horner form,
^{1}F: centered eval. at order 1,
^{2}F: centered eval. at order 2,

n	2	3	4	5
d	128	32	8	4
^{2}F	1028	18310	49647	104373
^{1}F	1594	47703	158076	298727
^{0}F	1916	102539	363274	576107

Nb of explored boxes and times in s., systems of n random dense pols of deg d

[Neu90] A. Neumaier.
Interval methods for systems of equations.
Cambridge University Press, 1990.

Evaluating large multivariate polynomials

Quickly: multivariate Horner scheme

Sharply: $\square F$ can be implemented by:

${}^0F$: Horner form,

${}^1F$: centered eval. at order 1, using n^2 first order derivatives

${}^2F$: centered eval. at order 2, using $\frac{n^2(n+1)}{2}$ second order derivatives

n	2		3		4		5	
d	128		32		8		4	
${}^2F$	1028	1.75s	18310	61.5s	49647	17.0s	104373	10.7s
${}^1F$	1594	2.24s	47703	107s	158076	36.4s	298727	21.9s
${}^0F$	1916	2.66s	102539	230s	363274	81.3s	576107	39.6s

Nb of explored boxes and times in s., systems of n random dense pols of deg d

[Neu90] A. Neumaier.
Interval methods for systems of equations.
Cambridge University Press, 1990.

Adapting arithmetic precision

Criteria of [Rev03]: $\{\mathbf{X}_1, \mathbf{X}_2\} = bisect(\mathbf{X})$

- $w(\mathbf{X}_1) \geq w(\mathbf{X})$ or $w(\mathbf{X}_2) \geq w(\mathbf{X})$
  $\rightarrow$ the width of **X** is near the machine ϵ
- $w(\Box F(\mathbf{X}_1)) \geq w(\Box F(\mathbf{X}))$ or $w(\Box F(\mathbf{X}_2)) \geq w(\Box F(\mathbf{X}))$
  $\rightarrow \Box F(\mathbf{X})$ is no more inclusion monotonic

[Rev03] N. Revol.
Interval newton iteration in multiple precision for the univariate case.
Numerical Algorithms, 34(2-4):417–426, 2003.

Adapting arithmetic precision

Criteria of [Rev03]: $\{\mathbf{X}_1, \mathbf{X}_2\} = bisect(\mathbf{X})$

- $w(\mathbf{X}_1) \geq w(\mathbf{X})$ or $w(\mathbf{X}_2) \geq w(\mathbf{X})$
  $\rightarrow$ the width of $\mathbf{X}$ is near the machine ϵ
- $w(\Box F(\mathbf{X}_1)) \geq w(\Box F(\mathbf{X}))$ or $w(\Box F(\mathbf{X}_2)) \geq w(\Box F(\mathbf{X}))$
  $\rightarrow \Box F(\mathbf{X})$ is no more inclusion monotonic

Heuristic criterion for Krawczyk operator:

$$K_F(\mathbf{X}) = \mathbf{P} - J_F(\mathbf{P})^{-1} F(\mathbf{P}) + \Box J_F(\ldots), \text{ where } \mathbf{P} \text{ is a point}$$

Certificate of existence and uniqueness only if $K_F(\mathbf{X}) \subset int(\mathbf{X})$

Adapting arithmetic precision

Criteria of [Rev03]: $\{\mathbf{X}_1, \mathbf{X}_2\} = \textit{bisect}(\mathbf{X})$

- $w(\mathbf{X}_1) \geq w(\mathbf{X})$ or $w(\mathbf{X}_2) \geq w(\mathbf{X})$
  $\rightarrow$ the width of $\mathbf{X}$ is near the machine ϵ
- $w(\Box F(\mathbf{X}_1)) \geq w(\Box F(\mathbf{X}))$ or $w(\Box F(\mathbf{X}_2)) \geq w(\Box F(\mathbf{X}))$
  $\rightarrow \Box F(\mathbf{X})$ is no more inclusion monotonic

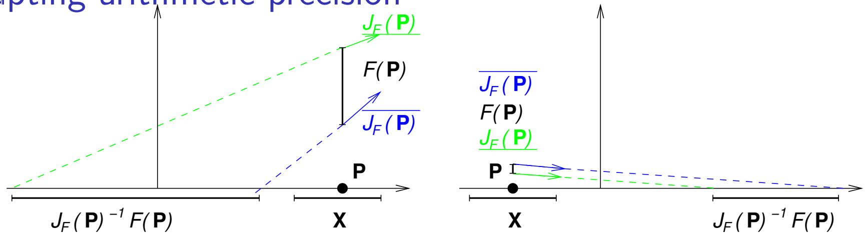
Heuristic criterion for Krawczyk operator:

$$K_F(\mathbf{X}) = \mathbf{P} - J_F(\mathbf{P})^{-1}F(\mathbf{P}) + \Box J_F(\ldots), \text{ where } \mathbf{P} \text{ is a point}$$

$$\Rightarrow w(K_F(\mathbf{X})) = \underbrace{w(\mathbf{P})}_{0} + \underbrace{w(J_F(\mathbf{P})^{-1}F(\mathbf{P}))}_{0} + w(\Box J_F(\ldots))$$

Certificate of existence and uniqueness only if $w(K_F(\mathbf{X})) < w(\mathbf{X})$

Adapting arithmetic precision



Heuristic criterion for Krawczyk operator:

$$K_F(\mathbf{X}) = \mathbf{P} - J_F(\mathbf{P})^{-1}F(\mathbf{P}) + \Box J_F(\ldots), \text{ where } \mathbf{P} \text{ is a point}$$

$$\Rightarrow w(K_F(\mathbf{X})) = \underbrace{w(\mathbf{P})}_{0} + \underbrace{w(J_F(\mathbf{P})^{-1}F(\mathbf{P}))}_{\geq 0} + w(\Box J_F(\ldots))$$

Certificate of existence and uniqueness only if $w(K_F(\mathbf{X})) < w(\mathbf{X})$

Arithmetic precision is increased for sub-boxes of $\mathbf{X}$ when:

$$w(J_F(\mathbf{P})^{-1}F(\mathbf{P})) \geq w(\mathbf{X}) \text{ and } w(F(\mathbf{P})) \geq w(\mathbf{X})$$

Adapting arithmetic precision

Example: Wilkinson polynomial with 15 roots

$$P(x) = (x-1)(x-2)\dots(x-10)\dots(x-15)$$

Initial domain:

$$\mathbf{x_0} = [9.999999999, 10.0000000001] \text{ width: } \simeq 1e-9$$

Initial precision: double (mantissa of 53 bits)

Without criterion: $\mathbf{x_0}$ bisected until machine ϵ is reached (619245 sub-boxes)

With criterion: precision is doubled, then $K_F(\mathbf{x_0}) \subset int(\mathbf{x_0})$

Results:

Datas: Random dense polynomials of degree d, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

[Imb16] Rémi Imbach.
A Subdivision Solver for Systems of Large Dense Polynomials.
Technical Report 476, INRIA Nancy, March 2016.

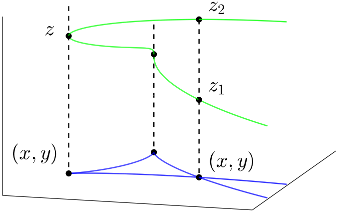
Numerical results: Isolating singularities of an apparent contour

system domain d	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$		
6	15	0.5		
7	105	4.44		
8	620	37.9		
9	3300	23.2		

means on 5 examples of sequential times.

Isolating singularities:

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$



Lemma 1: (x, y) is a node of $\mathcal{B} \Leftrightarrow (x, y, z_1, z_2)$ satisfies:

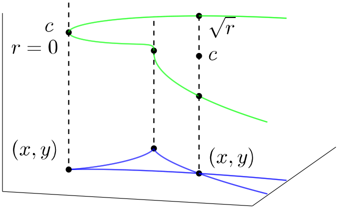
$$P(x, y, z_1) = Q(x, y, z_1) = P(x, y, z_2) = Q(x, y, z_2) = 0$$

Lemma 2: (x, y) is a cusp of $\mathcal{B} \Leftrightarrow (x, y, z)$ satisfies:

$$P(x, y, z) = Q(x, y, z) = P_z(x, y, z) = Q_z(x, y, z) = 0$$

Isolating singularities: the Ball system

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



c: center of z_1, z_2

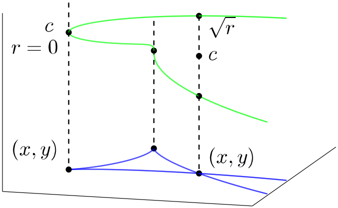
$r = \|cz_1\|_2^2$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4)\left\{\begin{array}{rl} \frac{1}{2}(P(x,y,c+\sqrt{r}) + P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r}) - P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r}) + Q(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r}) - Q(x,y,c-\sqrt{r})) & = 0 \end{array}\right.$$

Isolating singularities: the Ball system

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

when $r \to 0$

$$(\mathcal{S}_4) \begin{cases} P(x,y,c) &= 0 \\ P_z(x,y,c) &= 0 \\ Q(x,y,c) &= 0 \\ Q_z(x,y,c) &= 0 \end{cases}$$

Isolating singularities: the Ball system

[IMP15] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric and certified isolation of the singularities of the projection of a smooth space curve.
In *Proceedings of the 6th International Conferences on Mathematical Aspects of Computer and Information Sciences*, MACIS'15, 2015.

Lemma 4. Under some genericity assumptions, all the solutions of $\mathcal{S}_4$ in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$ are regular.

Lemma 3. Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4)\begin{cases} \frac{1}{2}(P(x,y,c+\sqrt{r})+P(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r})-P(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r})+Q(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r})-Q(x,y,c-\sqrt{r})) & =0 \end{cases}$$

Results:

Datas: Random dense polynomials of degree *d*, bit-size 8

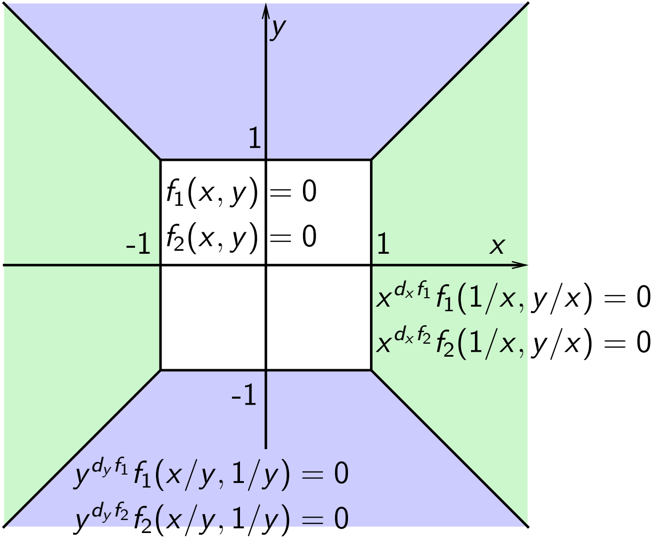
0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

[Imb16] Rémi Imbach.
A Subdivision Solver for Systems of Large Dense Polynomials.
Technical Report 476, INRIA Nancy, March 2016.

Numerical results: Isolating singularities of an apparent contour

system domain *d*	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$	$\mathcal{S}_4$, subd. $[-1,1]\times[-1,1]$
6	15	0.5	8.4
7	105	4.44	43.8
8	620	37.9	70.2
9	3300	23.2	45.6

means on 5 examples of sequential times.



[Neu90] A. Neumaier.
Interval methods for systems of equations.
Cambridge University Press, 1990.

Results:

Datas: Random dense polynomials of degree d, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

[Imb16] Rémi Imbach.
A Subdivision Solver for Systems of Large Dense Polynomials.
Technical Report 476, INRIA Nancy, March 2016.

Numerical results: Isolating singularities of an apparent contour

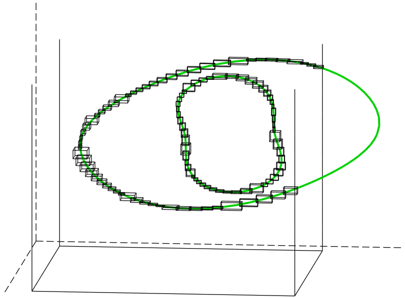
system	$\mathcal{S}_2$, RSCube	$\mathcal{S}_2$, subd.		$\mathcal{S}_4$, subd.	
domain	$\mathbb{R}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$
d					
6	15	0.5	1.35	8.4	11.3
7	105	4.44	124	43.8	54.2
8	620	37.9	57.7	70.2	99.2
9	3300	23.2	54.7	45.6	95.1

means on 5 examples of sequential times.

Computing the topology of $\mathcal{B}$: a geometric approach

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ such that

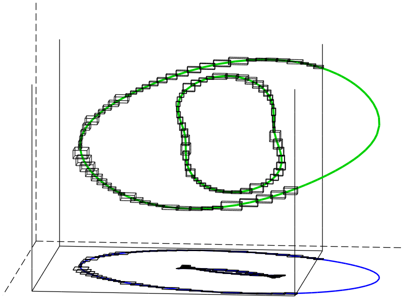
- $\mathcal{C} \subset \bigcup_k \mathbf{C}_k$,
- in each $\mathbf{C}_k$, $\mathcal{C} \cap \mathbf{C}_k$ is diffeomorphic to a close segment,
- each $\mathbf{C}_k$ has width less than η.



Computing the topology of $\mathcal{B}$: a geometric approach

Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$



Computing the topology of $\mathcal{B}$: a geometric approach

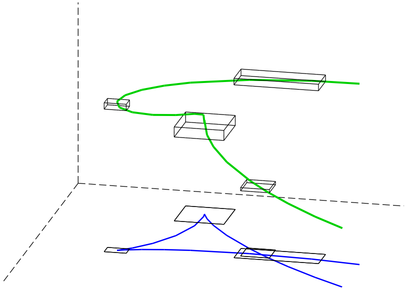
Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

Enclose $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

→ Isolate singularities:

- each cusp is in a $\mathbf{B}_k$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$

→ Singularities are in $\bigcup_k \mathbf{B}_k \cup \bigcup_{i,j} \mathbf{B}_{ij}$



Computing the topology of $\mathcal{B}$: a geometric approach

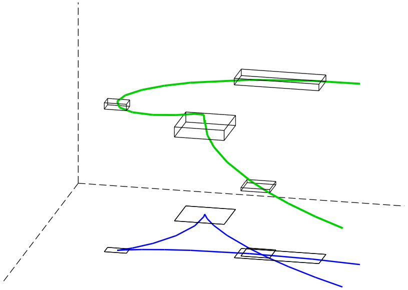
Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

Enclose $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

$\rightarrow$ Isolate singularities: $\mathcal{L}_c = \{\mathbf{B}_{p_1}, \ldots, \mathbf{B}_{p_{l_c}}\}$, $\mathcal{L}_n = \{\mathbf{B}_{q_1 r_1}, \ldots, \mathbf{B}_{q_{l_n} r_{l_n}}\}$

- each cusp is in a $\mathbf{B}_k$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$

$\rightarrow$ Singularities are in $\bigcup_k \mathbf{B}_k \cup \bigcup_{i,j} \mathbf{B}_{ij}$



Computing the topology of $\mathcal{B}$: a geometric approach

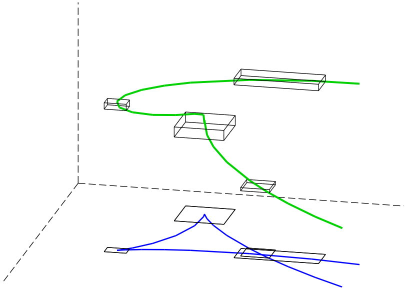
Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

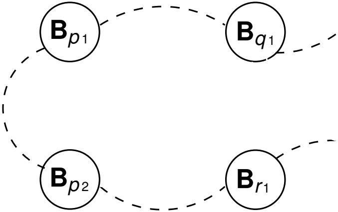
Enclose $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

Isolate singularities: $\mathcal{L}_c = \{\mathbf{B}_{p_1}, \ldots, \mathbf{B}_{p_{l_c}}\}$, $\mathcal{L}_n = \{\mathbf{B}_{q_1 r_1}, \ldots, \mathbf{B}_{q_{l_n} r_{l_n}}\}$

$\rightarrow$ Compute a graph:

- $\mathcal{G}_{\mathcal{B}} = (\{\mathbf{B}_k\}_{1\leq k\leq l}, \{(\mathbf{B}_k, \mathbf{B}_{k+1})\}_{1\leq k<l})$





Computing the topology of $\mathcal{B}$: a geometric approach

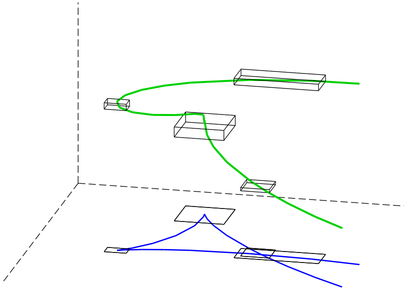
Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

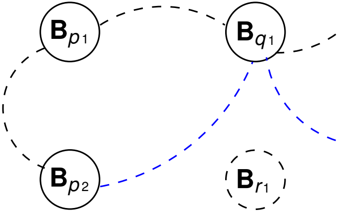
Enclose $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

Isolate singularities: $\mathcal{L}_c = \{\mathbf{B}_{p_1}, \ldots, \mathbf{B}_{p_{l_c}}\}$, $\mathcal{L}_n = \{\mathbf{B}_{q_1 r_1}, \ldots, \mathbf{B}_{q_{l_n} r_{l_n}}\}$

$\rightarrow$ Compute a graph:

- $\mathcal{G}_{\mathcal{B}} = (\{\mathbf{B}_k\}_{1\leq k\leq l}, \{(\mathbf{B}_k, \mathbf{B}_{k+1})\}_{1\leq k< l})$
- for each $\mathbf{B}_{q_1 r_1} \in \mathcal{L}_n$: identify $\mathbf{B}_{q_1}$ and $\mathbf{B}_{r_1}$





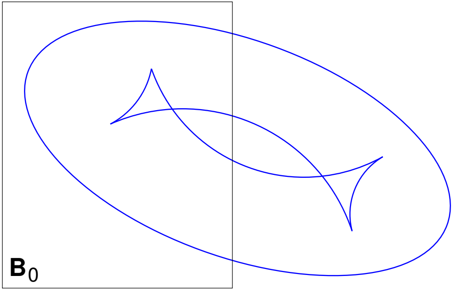
Computing the topology of $\mathcal{B}$: a geometric approach

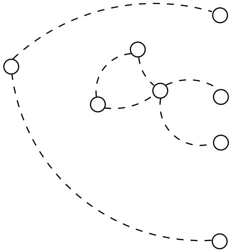
Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

Enclose $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

Isolate singularities: $\mathcal{L}_c = \{\mathbf{B}_{p_1}, \ldots, \mathbf{B}_{p_{l_c}}\}$, $\mathcal{L}_n = \{\mathbf{B}_{q_1 r_1}, \ldots, \mathbf{B}_{q_{l_n} r_{l_n}}\}$

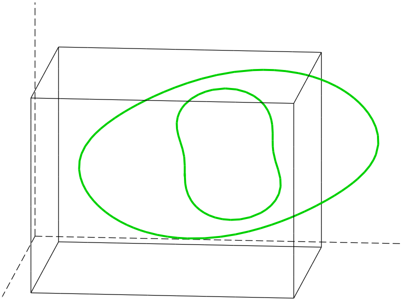
$\rightarrow$ Compute a graph: $\mathcal{G}_{\mathcal{B}}$ is homeomorphic to $\mathcal{B} \cap \mathbf{B}_0$





Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{C}_0$ a box of $\mathbb{R}^3$
$\mathcal{C} = \{C \in \mathbf{C}_0 | F(C) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$



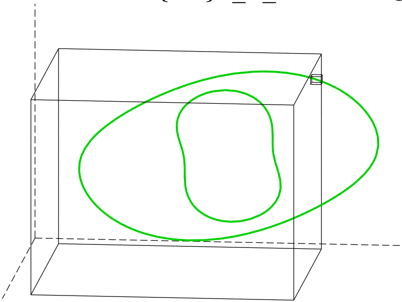
Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{C}_0$ a box of $\mathbb{R}^3$
$\mathcal{C} = \{C \in \mathbf{C}_0 | F(C) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Certified path-tracker:

Input: $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{C}_0$ box of $\mathbb{R}^3$, $\epsilon \in \mathbb{R}^+_*$
An initial box $\mathbf{C} \in \mathcal{C}^i$

Output: a sequence of boxes $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ enclosing $\mathcal{C}^i$.



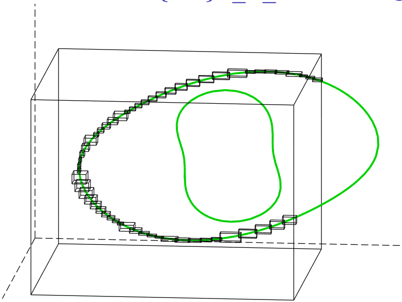
Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{C}_0$ a box of $\mathbb{R}^3$
$\mathcal{C} = \{C \in \mathbf{C}_0 | F(C) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \dots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Certified path-tracker:

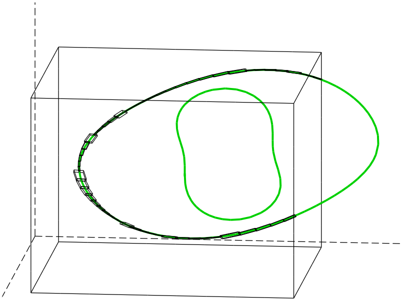
Input: $F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{C}_0$ box of $\mathbb{R}^3$, $\epsilon \in \mathbb{R}^+_*$
An initial box $\mathbf{C} \in \mathcal{C}^i$

Output: a sequence of boxes $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ enclosing $\mathcal{C}^i$.



Certified numerical tools: 1-dim solver

[MGGJ13] B. Martin, A. Goldsztejn, L. Granvilliers, and C. Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

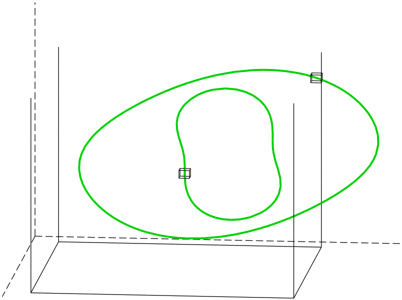


Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{B}_0$ a box of $\mathbb{R}^2$
$\mathcal{C} = \{C \in \mathbf{B}_0 \times \mathbb{R} | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$
(A1) holds for generic polynomials P, Q

Finding one point on each connected component

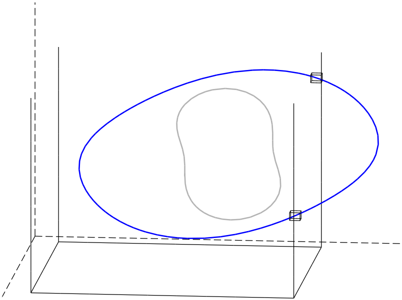


Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points

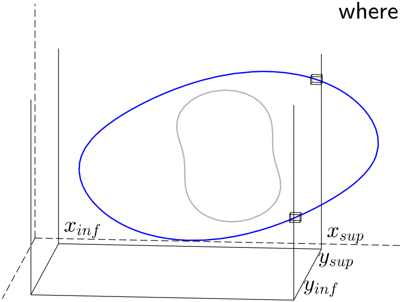


Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0,1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points



$\mathcal{C} \cap (\partial\mathbf{B}_0 \times \mathbb{R})$ are the solutions of the 4 systems:

$$\begin{cases} P(x=a,y,z)=0 \\ Q(x=a,y,z)=0 \end{cases}$$

$$\begin{cases} P(x,y=b,z)=0 \\ Q(x,y=b,z)=0 \end{cases}$$

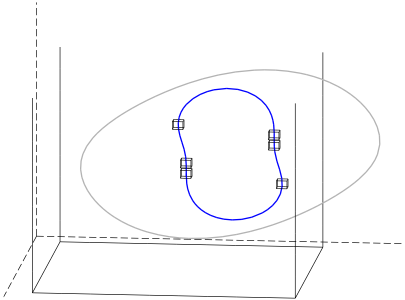
where $a \in \{x_{inf}, x_{sup}\}$,
$b \in \{y_{inf}, y_{sup}\}$

Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points



Finding one point on each connected component

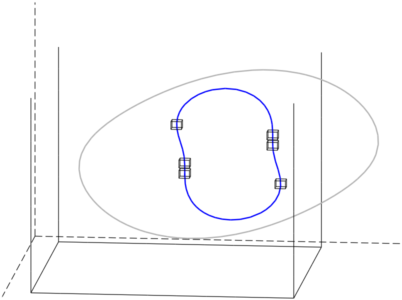
Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points

x-critical points of $\mathcal{C}$ are the solutions of the system:

$$\begin{cases} P(x,y,z) &= 0 \\ Q(x,y,z) &= 0 \\ \begin{vmatrix} P_y & P_z \\ Q_y & Q_z \end{vmatrix}(x,y,z) &= 0 \end{cases}$$



Results:

Datas: Random dense polynomials of degree *d*, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`

Path tracker: prototype in `python/cython`

[MGGJ13] B. Martin, A. Goldsztejn, L. Granvilliers, and C. Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

Numerical results: Isolating singularities of an apparent contour

system domain *d*	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$	$\mathcal{S}_4$, subd. $[-1,1]\times[-1,1]$	with $\mathcal{C}$ $[-1,1]\times[-1,1]$
6	15	0.5	8.4	2.36
7	105	4.44	43.8	4.13
8	620	37.9	70.2	5.91
9	3300	23.2	45.6	5.30

means on 5 examples of sequential times.