Exact solution to a parametric linear programming problem

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Parametric linear programming (PLP) problem

$$f(x,p) = c^T(p)x(p),$$

where $c_i(p)$ are nonlinear functions of p, and constraint given as a linear interval parametric (LIP) system

$$A(p)x(p) = b(p), p \in \mathbf{p}$$

where $a_{ij}(p)$, $b_i(p)$ are affine-linear functions of p.

Goal: determine the range

$$\mathbf{f}^*(A(p), b(p), c(p), \mathbf{p}) = \square\{f(x,p) : A(p)x = b(p), p \in \mathbf{p}\}.$$

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The endpoints $\underline{f}^*$ and $\overline{f}^*$ of the range $\mathbf{f}^*$ can be determined as the global solutions of the following two optimization problems:

$$\underline{f}^* = \min\left\{f(x,p) : A(p)x = b(p), p \in \mathbf{p}\right\},$$

$$\underline{f}^* = \max\left\{f(x,p) : A(p)x = b(p), p \in \mathbf{p}\right\}.$$

[2] Lubomir Kolev, A class of iterative methods for determining p-solutions of linear interval parametric systems. Reliable Computing, 2016, vol. 22, pp. 26-46.

Iterative method

$\nu = 0$, $\mathbf{p}^{(\nu)} = \mathbf{p}$
while $\mathrm{maxr}(\mathbf{p}^{(\nu)}) \geqslant \varepsilon_p$ **do**
  Find in $\mathbf{p}^{(\nu)}$ an interval $\mathbf{f}^u$, which encloses $\underline{f}^*$
  Using $\mathbf{f}^u$ and a constraint equation, reduce the domain $\mathbf{p}^{(\nu)}$ to a narrower domain $\mathbf{p}^{(\nu+1)}$
  if $q(\mathbf{p}^{(\nu)}, \mathbf{p}^{(\nu+1)}) > \varepsilon_q$ *then* $\nu = \nu + 1$ **else**
  return Only a crude estimation of $\underline{f}^*$ has been found
end while
return $\underline{p}^*$ providing $\underline{f}^*$ has been found to lie within $\mathbf{p}^{(\nu)}$

Linear form of the parametrized solution (p-solution) of LIP

$$\mathbf{x}(p) = Lp + \mathbf{a}, p \in \mathbf{p},$$

where L is a real $n \times m$ matrix and $\mathbf{a} = x^c + s[-1, 1]$ is an n-dimensional interval vector.

[3] Kolev L. (2014) Parameterized solution of linear interval parametric systems. Applied Mathematics and Computation 246: 229-246

Quadratic form of the p-solution

$$\mathbf{x}(p) = Q\theta(p) + Lp + \mathbf{a}, p \in \mathbf{p},$$

where Q is a three dimensional $n \times m \times m$ array, $\theta_j(p) = p_j^2$, and L and $\mathbf{a}$ have the same meaning as before.

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$$\mathbf{f}(p) = f^0 + \sum_{j=1}^{m} L_j^0 p_j + s^0[-1,1], p \in \mathbf{p},$$

where $f^0 = \sum_{i=1}^{n} c_i x_i^c$, $L_j^0 = \sum_{i=1}^{n} c_i L_{ij}$, $s^0 = \sum_{i=1}^{n} |c_i| s_i$.

Computing $\mathbf{f}^u$ enclosing $\underline{\mathbf{f}}^*$

$$\underline{\mathbf{f}}^* \in \mathbf{f}^u = \underline{\lambda} + \mathbf{g},$$

where $\underline{\lambda} = -\sum_{j=1}^{m} |L_j^0|$, and $\mathbf{g} = f^0 + s^0[-1,1]$.

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Constraint equation

$$\sum_{j=1}^{m} L_j^0 p_j + s = d,$$

where $p_{\in}\mathbf{p_j}$, $s \in s^0[-1,1]$, $d \in s^0[-1,1] - \sum_{j=1}^{m} |L_j^0|$.

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Constraint propagation to narrow $\mathbf{p}$

We select the index i, which corresponds to the maximum component $|L_j^0|$, $j = 1, \ldots, m$.

$$p_i = b / L_i^0,$$

where $b = d - \sum_{j \neq i} L_j^0 p_j + s$.

Contracted domain

$$\mathbf{p}' = \mathbf{p} \cap (\mathbf{b} / L_i^0)$$

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Example. We consider a special case of PLP with $n = 3$ and $c = (1, 1, 1)^T$:

$$f(x, p) = \sum_{i=1}^{3} x_i(p),$$

subject to $A(p)x = b(p)$, $p \in \mathbf{p}$, where

$$A(p) = \begin{bmatrix} p_1 & p_2 + 1 & -p_3 \\ -p_2 & -3 & p_1 \\ 2 - p_3 & 4p_2 + 1 & 1 \end{bmatrix}, \quad b(p) = \begin{bmatrix} 2p_1 \\ p_3 - 1 \\ -1 \end{bmatrix}.$$

The parameter vectors are of the form

$$\mathbf{p}(\rho) = p^c + \rho[-p^\Delta, p^\Delta],$$

where $p^c = (0.5, 0.5, 0.5)^T$, $p^\Delta = (0.5, 0.5, 0.5)^T$, and ρ is a variable.

Table: Results for PLP ($\varepsilon_p = \varepsilon_q = 1.0e^{-6}$)

	inf	sup	inf			sup		
			p_1	p_2	p_3	p_1	p_2	p_3
0.02	-1.256	-1.219	0.510	0.510	0.490	0.490	0.490	0.510
0.05	-1.285	-1.193	0.525	0.525	0.475	0.475	0.475	0.525
0.1	-1.336	-1.152	0.550	0.550	0.450	0.450	0.450	0.550
0.15	-1.390	-1.114	0.575	0.575	0.425	0.425	0.425	0.575
0.17	-1.413	-1.099	0.585	0.585	0.415	0.415	0.415	0.585
0.18	-1.424	-1.092	0.590	0.590	0.410	0.410	0.410	0.590
0.2	-2.015	-0.463	**p**	**p**	**p**	**p**	**p**	**p**

If the method fail (only crude bounds are produced), then we can still obtain relatively good bounds on $\mathbf{f}^*$ from the parametric solution and the equation:

$$\mathbf{f}^* \subseteq \mathbf{f} = f_i^0 + \sum_{j=1}^{m} |L_j^0|[-1,1] + s^0[-1,1].$$

Table: Bounds on $\mathbf{l}^*$

	Exact solution		Iterative method	
0.3	-1.5736	-1.0177	-1.8473	-0.6343
0.4	-1.7146	-0.9542	-2.3684	-0.1181
0.5	-1.8724	-0.8853	-3.3717	0.8779

- The proposed iterative method yields sharp bound on the exact solution to a parametric linear programming problem assuming that the initial intervals are relatively narrow.
- The limitations can be overcome by using some more sophisticated methods for computing the upper and lower bound on the exact bounds or using some more sophisticated constraint satisfaction technique.
- We can also use better methods for computing the p-solution, for example using quadratic approximation.
- It is possible extend the prosposed approach to problems with nonlinear dependencies in the constraints system.
- The p-solution has many potentials: can be used to computed the hull solution of parametric interval linear systems,