Inner approximation of a capture set

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Capture set

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Definition. Given the state equation $\dot{x} = f(x)$. Let $\varphi$ be the flow map. The *capture* set of the target $T \subset \mathbb{R}^n$ is:

$$\overline{T} = \{x_0 \mid \exists t \geq 0, \varphi(t, x_0) \in T\}.$$
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An *interval* is a *domain* which encloses a real number.

A *polygon* is a *domain* which encloses a vector of $\mathbb{R}^n$.

A *labyrinth* is a *domain* which encloses a path.
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Labyrinths can be made more accurate by adding polygons.
Or using doors instead of a graph
Here, a **labyrinth** $\mathcal{L}$ is composed of
- A paving $\mathcal{P}$
- A polygon for each box of $\mathcal{P}$
- Doors between adjacent boxes
The set of labyrinths forms a lattice with respect to $\subset$. $\mathcal{L}_a \subset \mathcal{L}_b$ means:

- the boxes of $\mathcal{L}_a$ are subboxes of the boxes of $\mathcal{L}_b$.
- The polygons of $\mathcal{L}_a$ are included in those of $\mathcal{L}_b$.
- The doors of $\mathcal{L}_a$ are thinner than those of $\mathcal{L}_b$. 
Note that yellow polygons are convex.
Inner approximation of a capture set
**Main idea:** Compute an outer approximation of the complementary of $\overleftarrow{T}$:

$$\overleftarrow{T} = \{x_0 \mid \forall t \geq 0, \varphi(t, x_0) \notin T\}$$

Thus, we search for a path that never reach $T$. 
Target contractor. If a box $[x]$ of $\mathcal{P}$ is included in $T$ then remove $[x]$ and close all doors entering in $[x]$. 
Flow contractor. For each box $[x]$ of $\mathcal{P}$, we contract the polygon using the constraint $\dot{x} = f(x)$. 

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Inner propagation
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Outer propagation
An interpretation can be given only when the fixed point is reached.
Car on the hill

Inner approximation of a capture set
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 9.81 \sin \left( \frac{11}{24} \cdot \sin x_1 + 0.6 \cdot \sin(1.1 \cdot x_1) \right) - 0.7 \cdot x_2
\end{align*}
\]
Research box $X_0 = [-1, 13] \times [-10, 10]$
Blue: $T_{out} = X_0$; Red: $T_{in} = [2, 9] \times [-1, 1]$
Combined with an outer propagation
Van der Pol system
Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (1 - x_1^2) \cdot x_2 - x_1
\end{align*}
\]

and the box \( X_0 = [-4, 4] \times [-4, 4] \).
\( f \rightarrow -f \); \( T = \overline{X_0} \cup [-0.1, 0.1]^2 \).
\( f \rightarrow -f \; ; \; \mathbb{T}_{out} = \overline{X} ; \; \mathbb{T}_{in} = [0.5, 1]^2. \)
Combined with an outer propagation
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