Robust Hybrid State Estimation using Interval Methods

Nacim RAMDANI

University of Orléans, PRISME EA 4229, Bourges

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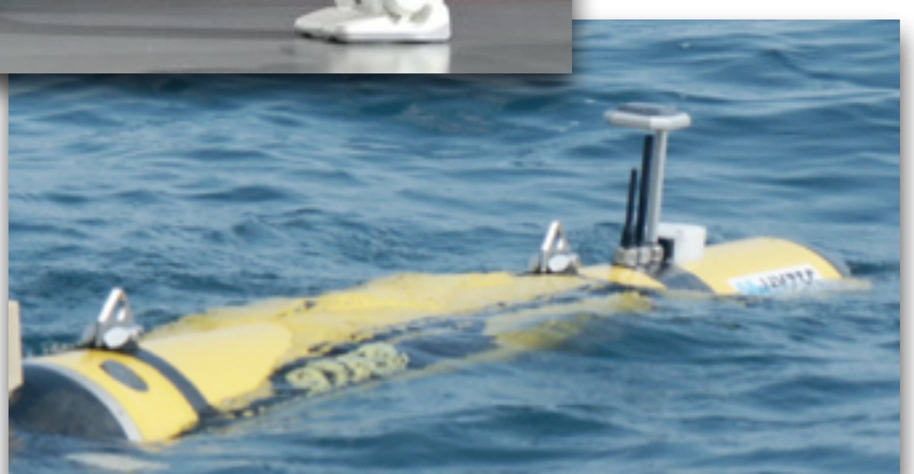
*co-supervised with Louise TRAVE-MASSUYES (**LAAS-CNRS, Toulouse**)*

SWIM 2016 @ ENS Lyon

Outline

- **Hybrid dynamical systems**
- Set membership estimation
- Hybrid reachability based approach
- Example
- Research directions

Hybrid Cyber-Physical Systems

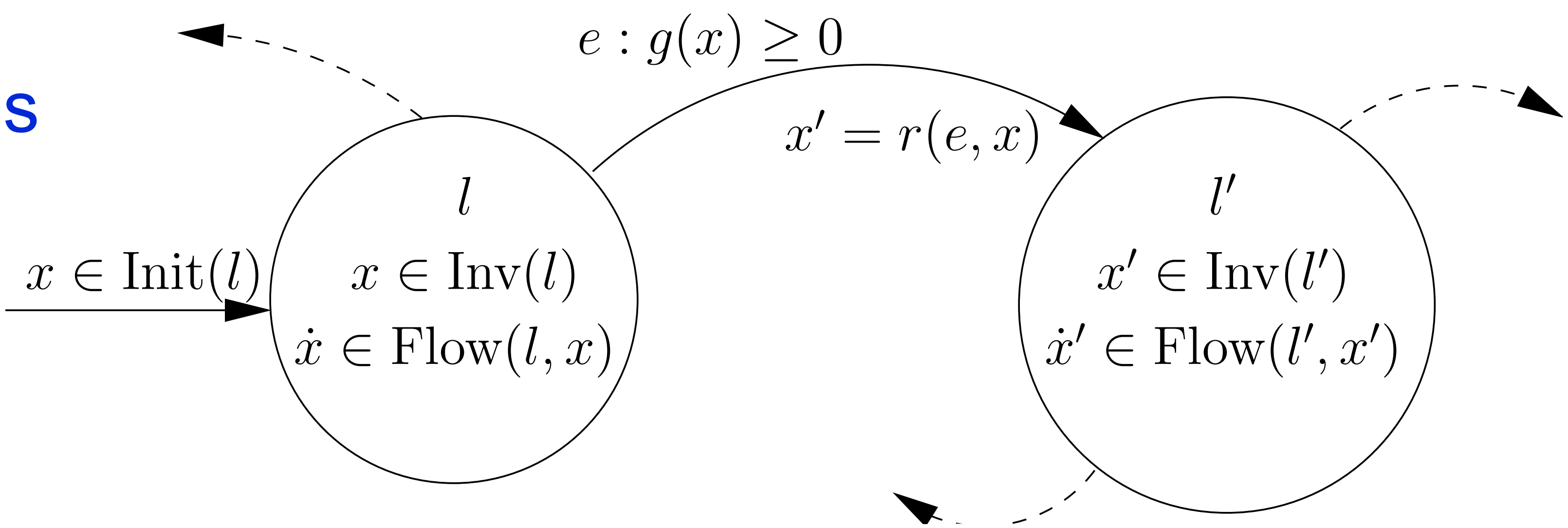


- **Interaction discrete + continuous dynamics**
- **Safety-critical embedded systems**
- **Networked autonomous systems**

Hybrid Cyber-Physical Systems

Modelling → hybrid automaton (Alur, et al. 1995)

- Non-linear continuous dynamics
- Nonlinear guards sets
- Nonlinear reset functions
- Bounded uncertainty



$$H = (\mathcal{Q}, \mathcal{D}, \mathcal{P}, \Sigma, \mathcal{A}, \mathsf{Inv}, \mathcal{F}),$$

Continuous dynamics

$$\begin{aligned} \mathsf{flow}(q) :& \quad \dot{\mathbf{x}}(t) = f_q(\mathbf{x}, \mathbf{p}, t), \\ \mathsf{Inv}(q) :& \quad \nu_q(\mathbf{x}(t), \mathbf{p}, t) < 0, \end{aligned}$$

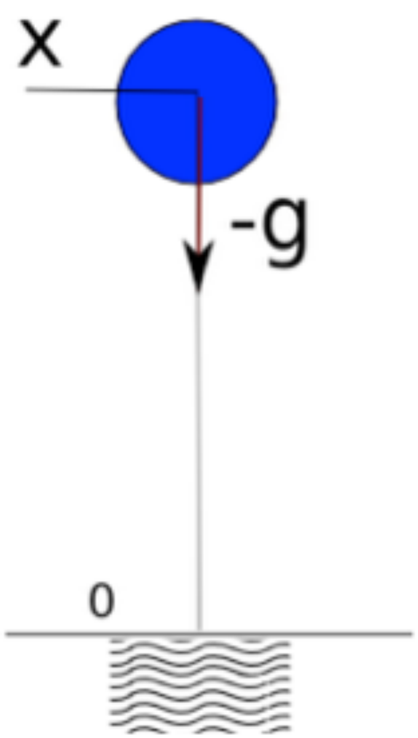
Discrete dynamics

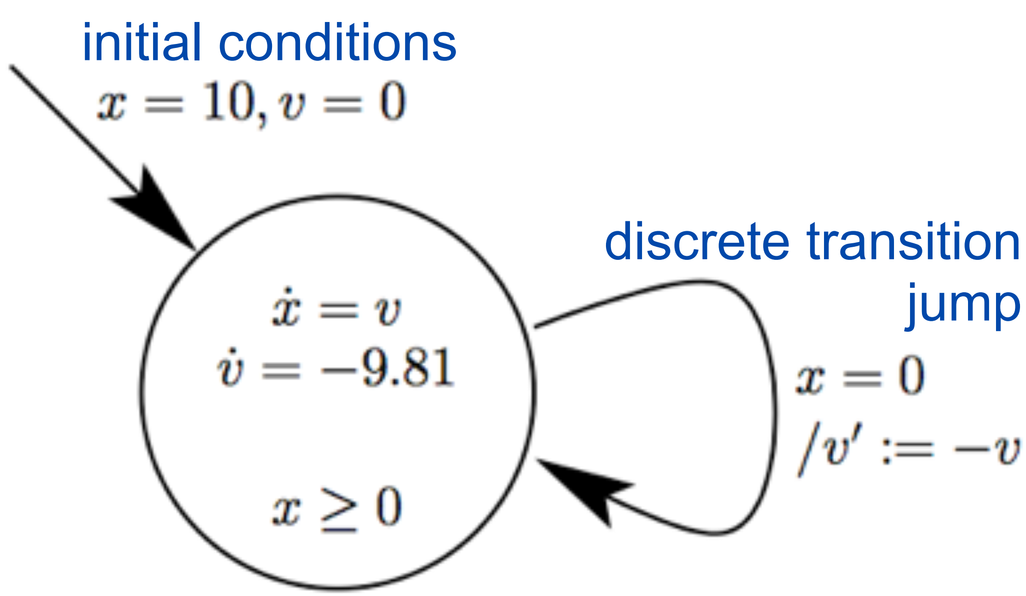
$$\begin{aligned} \mathcal{A} \ni e :& \quad (q \to q') = (q, \mathsf{guard}, \sigma, \rho, q'), \\ \mathsf{guard}(e) :& \quad \gamma_e(\mathbf{x}(t), \mathbf{p}, t) = 0, \end{aligned}$$

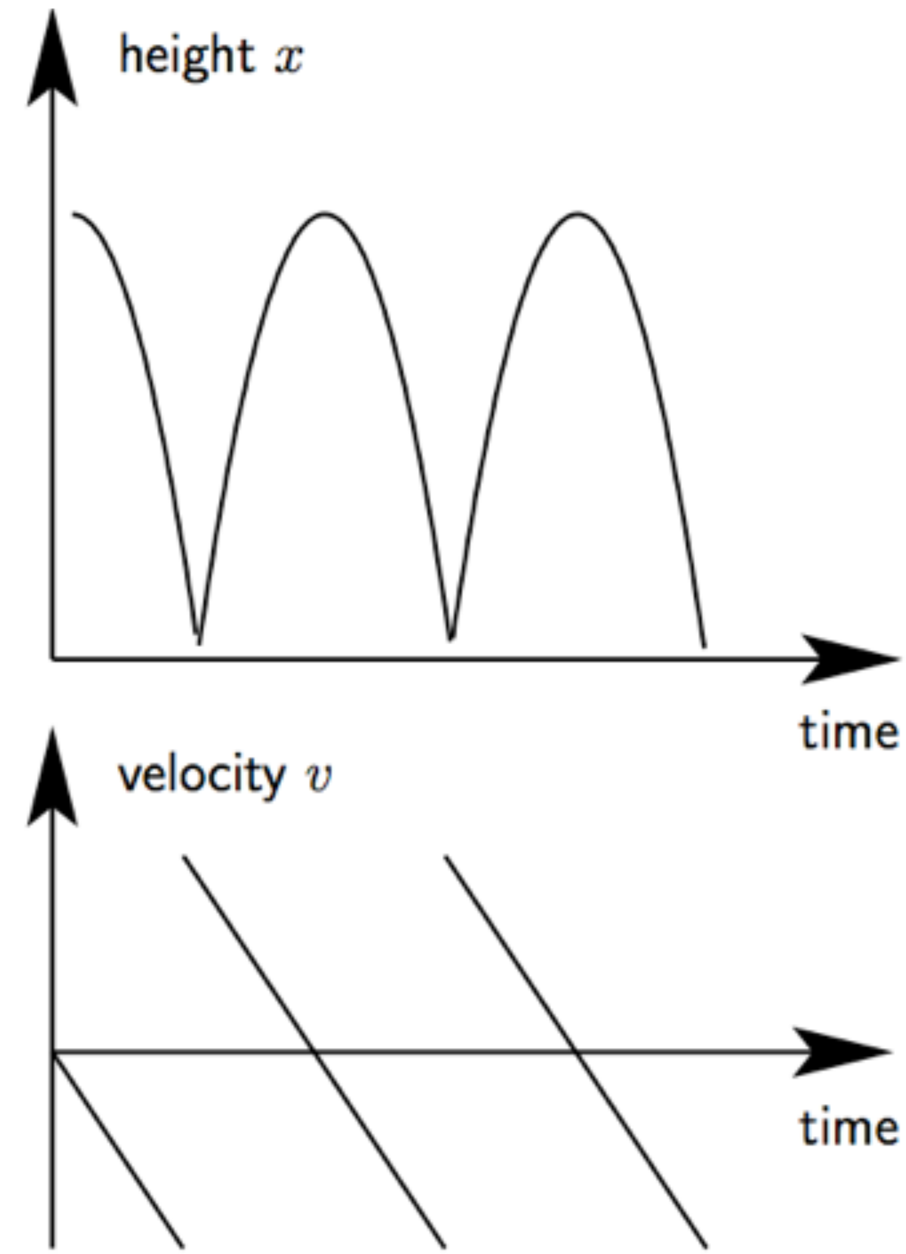
$$t_0 \leq t \leq t_N, \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \subseteq \mathbb{R}^n, \quad \mathbf{p} \in \mathbb{P}$$

Hybrid Cyber-Physical Systems

Example : the bouncing ball

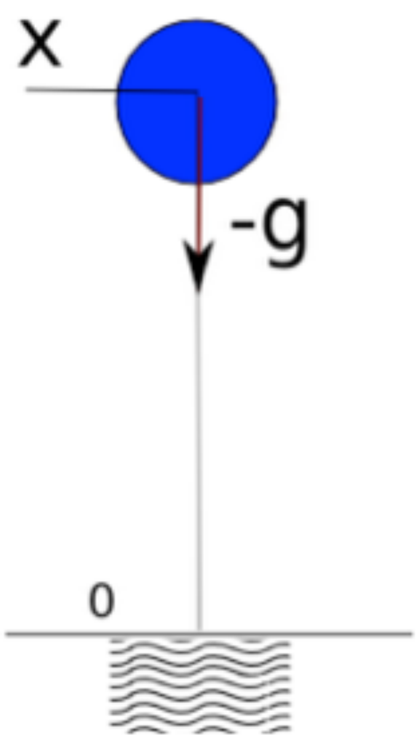


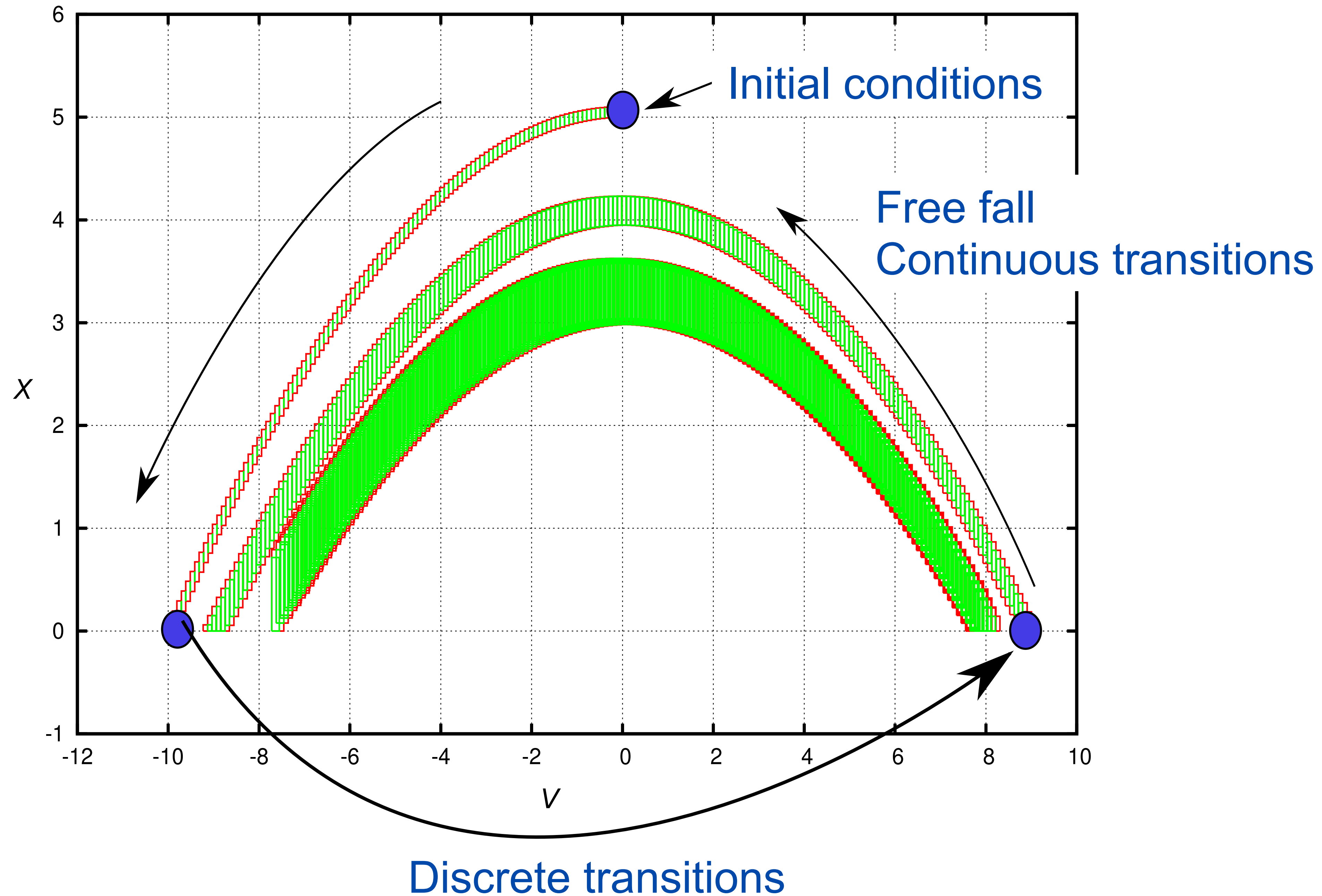




Hybrid Cyber-Physical Systems

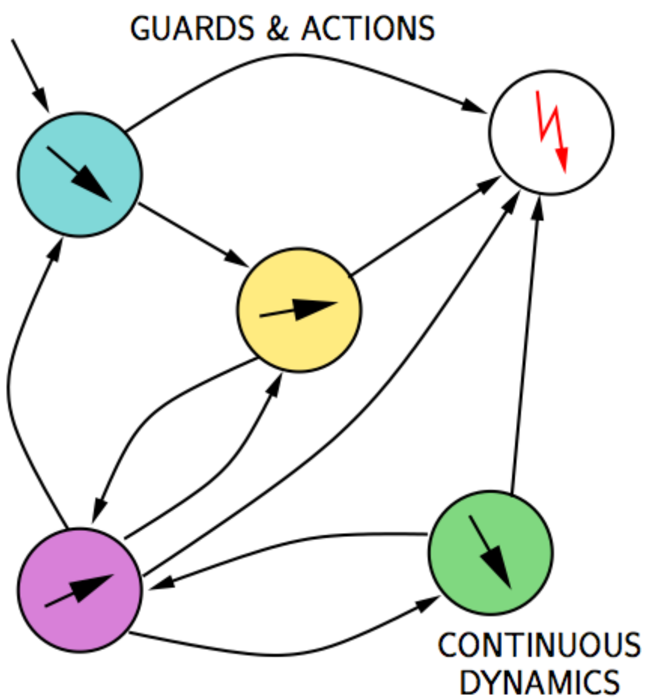
- **Example :** the bouncing ball



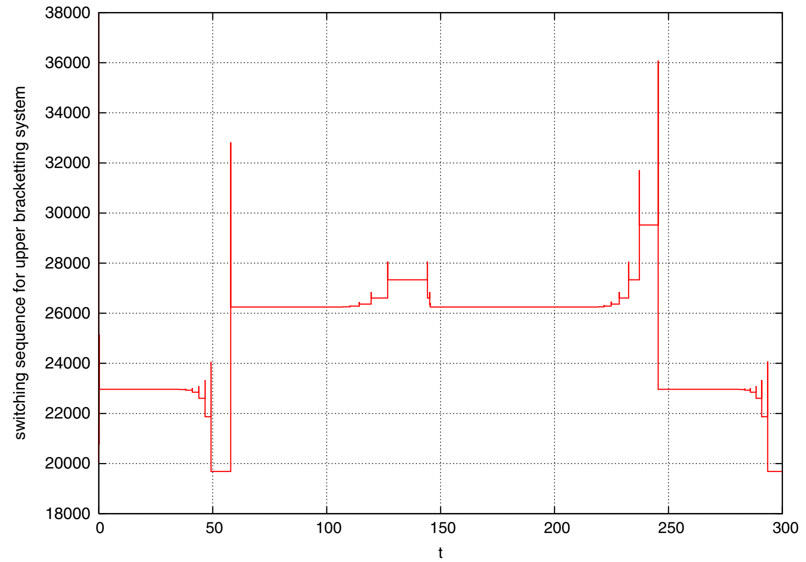


Estimation of Hybrid State

- **Modelling → hybrid automaton**
  - Nonlinear …
  - Bounded uncertainty



- **Hybrid State Estimation → reconstruct system variables**
  - Switching sequence
  - Continuous variables
  - **Hybrid solution trajectory tube**

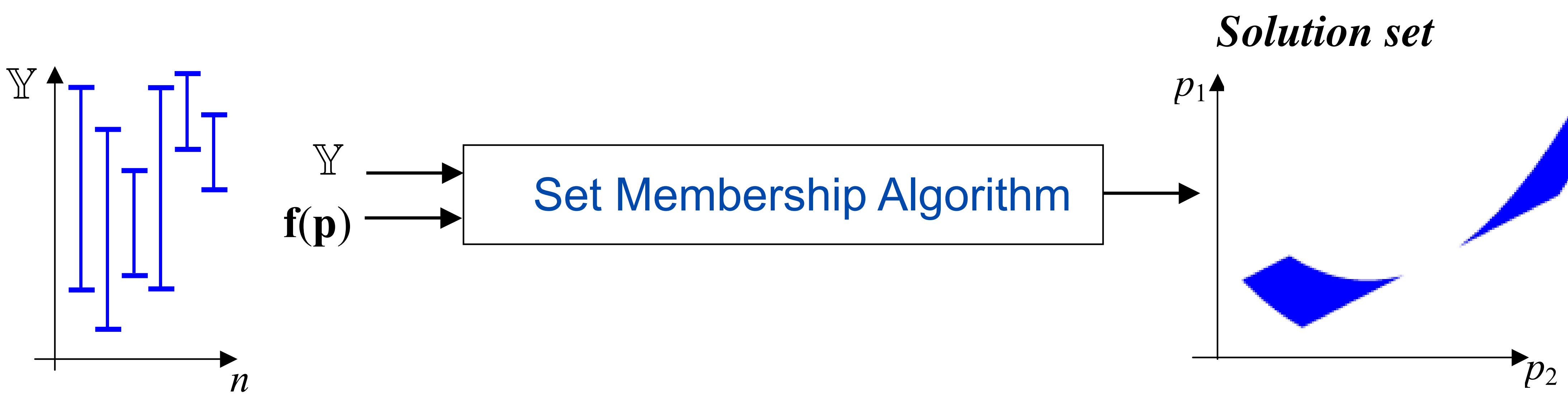


Outline

- Hybrid dynamical systems
- **Set membership estimation**
- Hybrid reachability based approach
- Example
- Research directions

Set Membership Estimation

- Unknown but bounded-error framework



Hypothesis

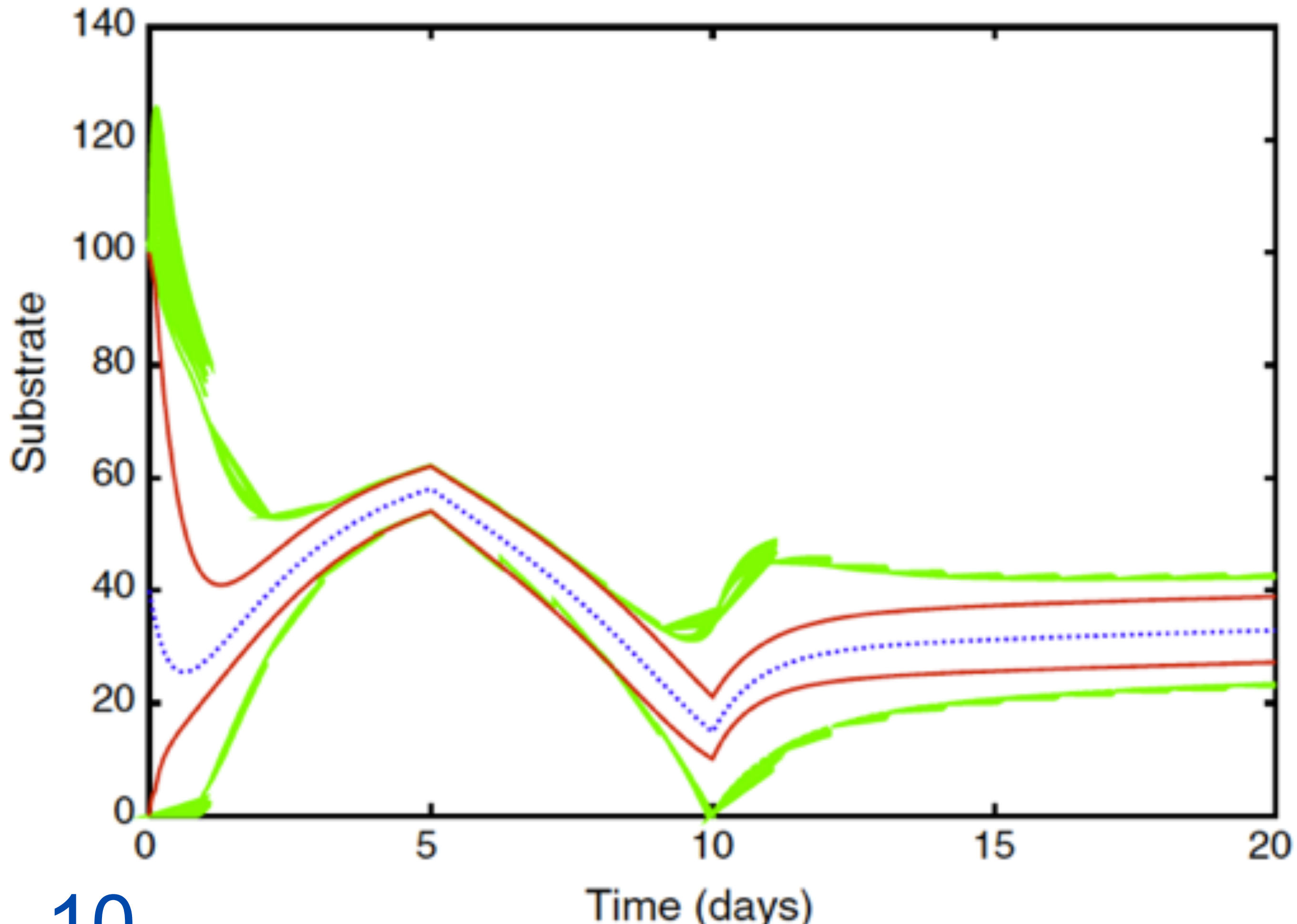
Uncertainties and errors are bounded with known prior bounds

A set of feasible solutions

$$\mathbb{S} = \{\mathbf{p} \in \mathbb{P} | \mathbf{f}(\mathbf{p}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}) \cap \mathbb{P}$$

Set Membership Estimation

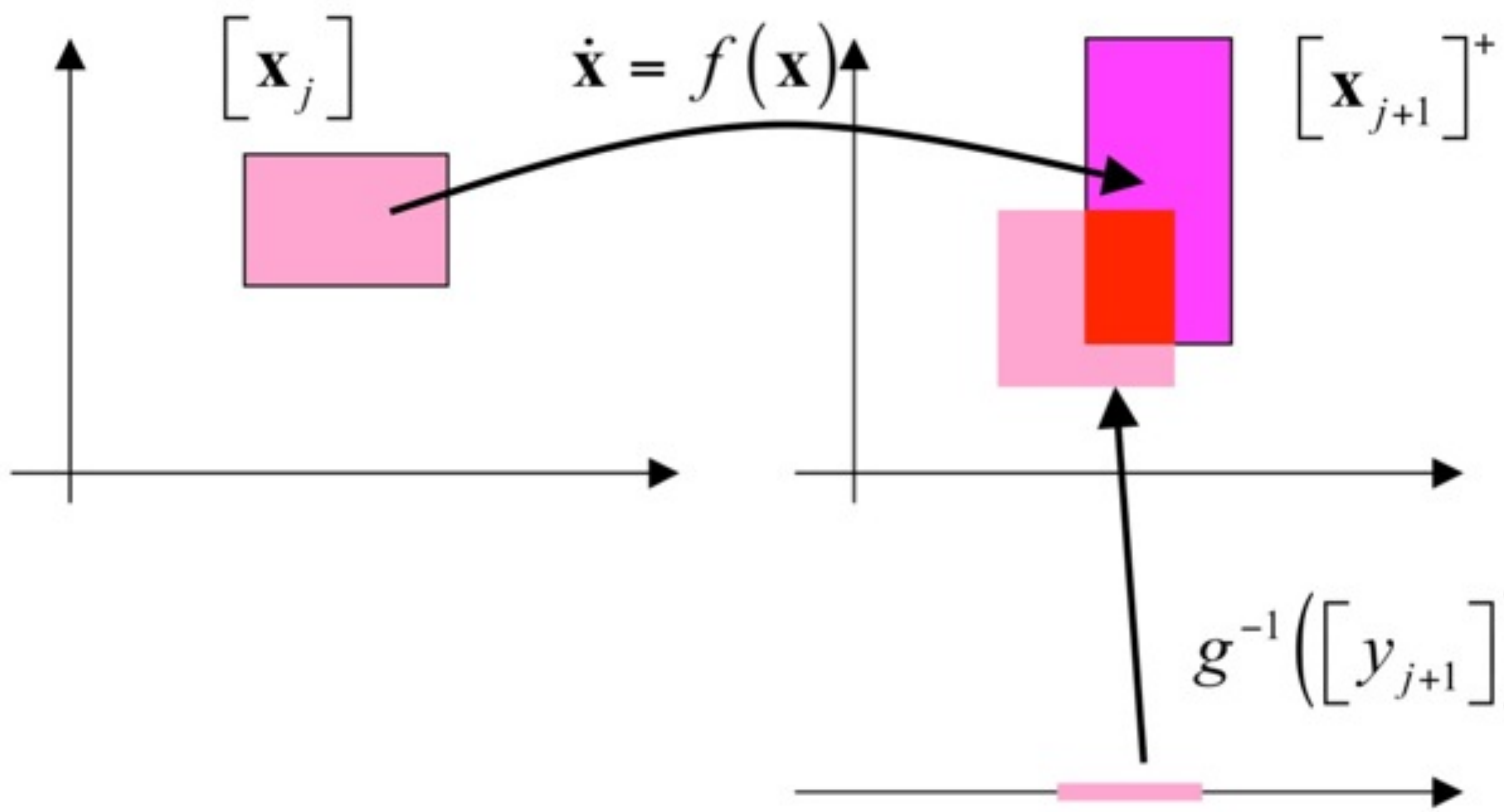
- **State estimation with continuous systems**
  - **Interval observers**
    - (Moisan, et al. 2009), (Mazenc & Bernard, 2010), (Meslem & Ramdani, 2011), (Raïssi, et al., 2012), (Combastel, 2013), (El Thabet, et al. 2014), (Efimov, et al. 2015)
    - Monotonicity
    - Change of coordinates
    - LMI ….
    - Ensure practical stability



Set Membership Estimation

State estimation with continuous systems

- **Prediction - Correction / Filtering approaches**
  - (Raïssi et al., 2005), (Meslem, et al, 2010), (Milanese & Novara, 2011), (Kieffer & Walter, 2011) …
  - Reachability + Set inversion
  - Forward backward consistency



Set Membership Estimation

Set inversion. Parameter estimation

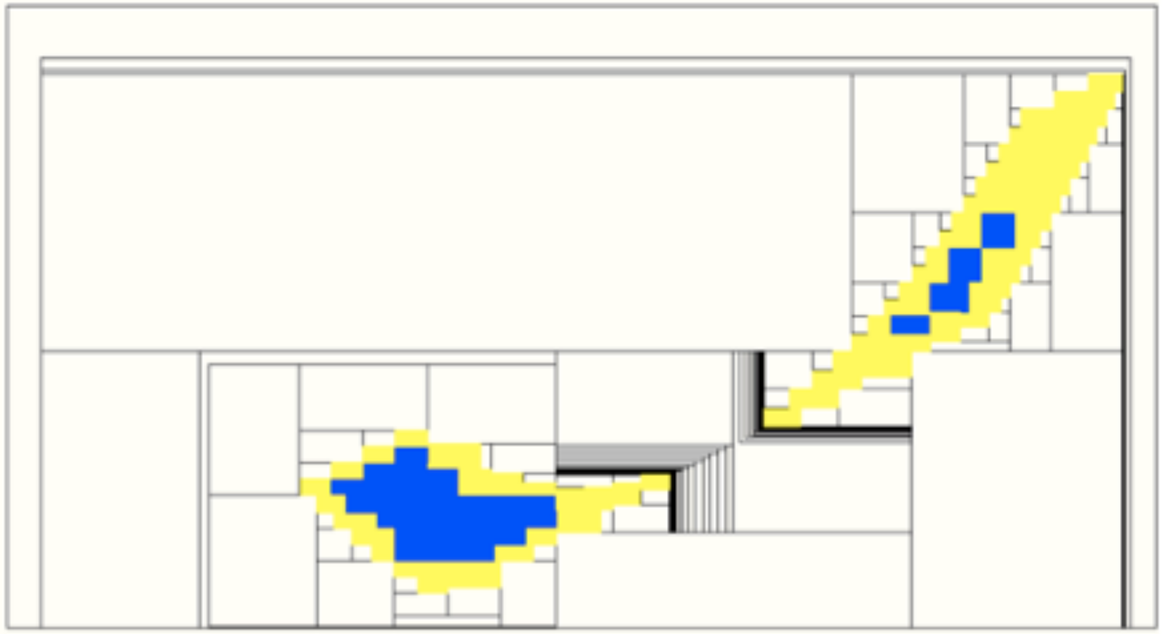
- Branch-&-bound, branch-&-prune, interval contractors …
  (Jaulin, et al. 93) (Raïssi et al., 2004)

$$\mathbb{S} = \{\mathbf{z} \in \mathcal{Z}, \mid f(\mathbf{z}) \in \mathcal{Y}\} \quad \to \underline{\mathbb{S}} \subseteq \mathbb{S} \subseteq \overline{\mathbb{S}}$$

$f([\mathbf{z}]) \subseteq \mathcal{Y}$ $\Rightarrow [\mathbf{z}] \subseteq \underline{\mathbb{S}}$: inner approximation

$f([\mathbf{z}]) \cap \mathcal{Y} = \emptyset$ $\Rightarrow [\mathbf{z}] \not\subseteq \overline{\mathbb{S}}$: outer approximation $\Rightarrow [\mathbf{z}] \subseteq \mathcal{Z} \backslash \overline{\mathbb{S}}$

otherwise partition …



Set Membership Estimation

- State estimation with Continuous systems
  - Interval observers
  - Prediction-correction / Filtering approaches
- **State estimation with Hybrid systems**
  - Piecewise affine systems (Bemporad, et al. 2005)
  - ODE + CSP (Goldsztejn, et al., 2010)
  - Nonlinear case (Benazera & Travé-Massuyès, 2009)
  - SAT mod ODE (Eggers, Ramdani, et al., 2012)
  - **Reachability-based (Maïga, Ramdani, et al. 2015)**.

Outline

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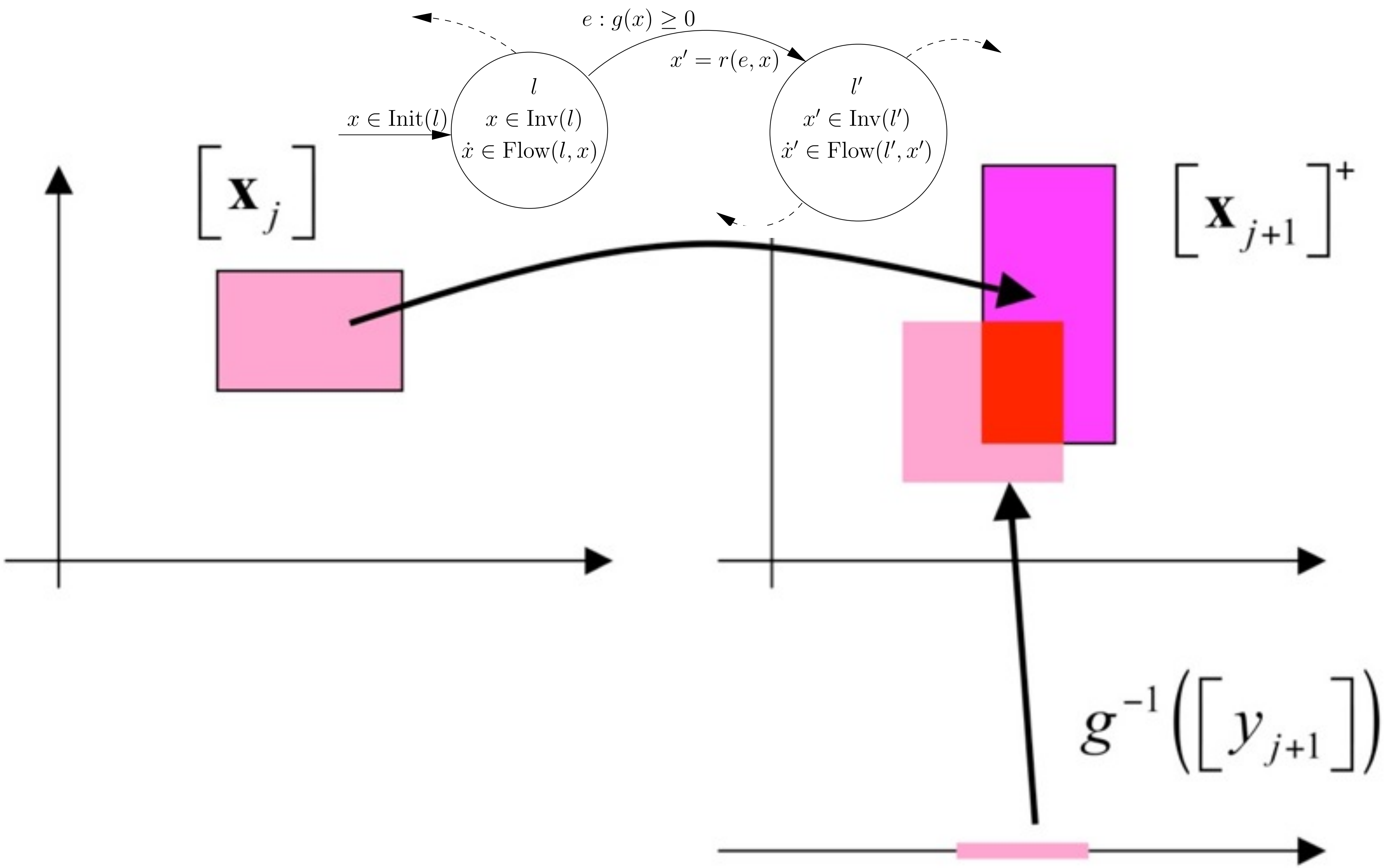
Reachability based approach

- **Predictor-Corrector** approach for hybrid systems

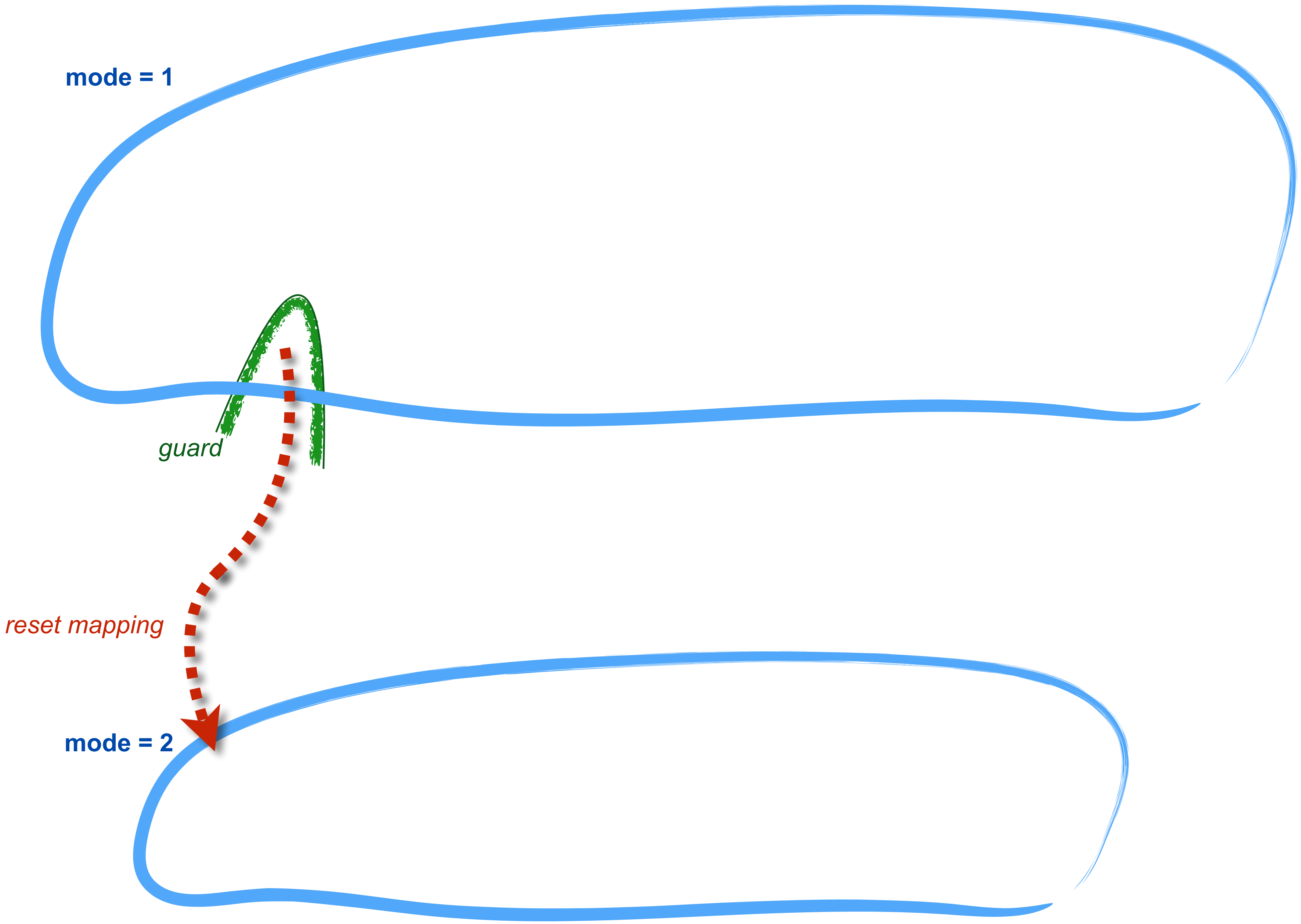
Reachability based approach

- **Predictor-Corrector** approach for hybrid systems



Reachability based approach

Reachability based approach



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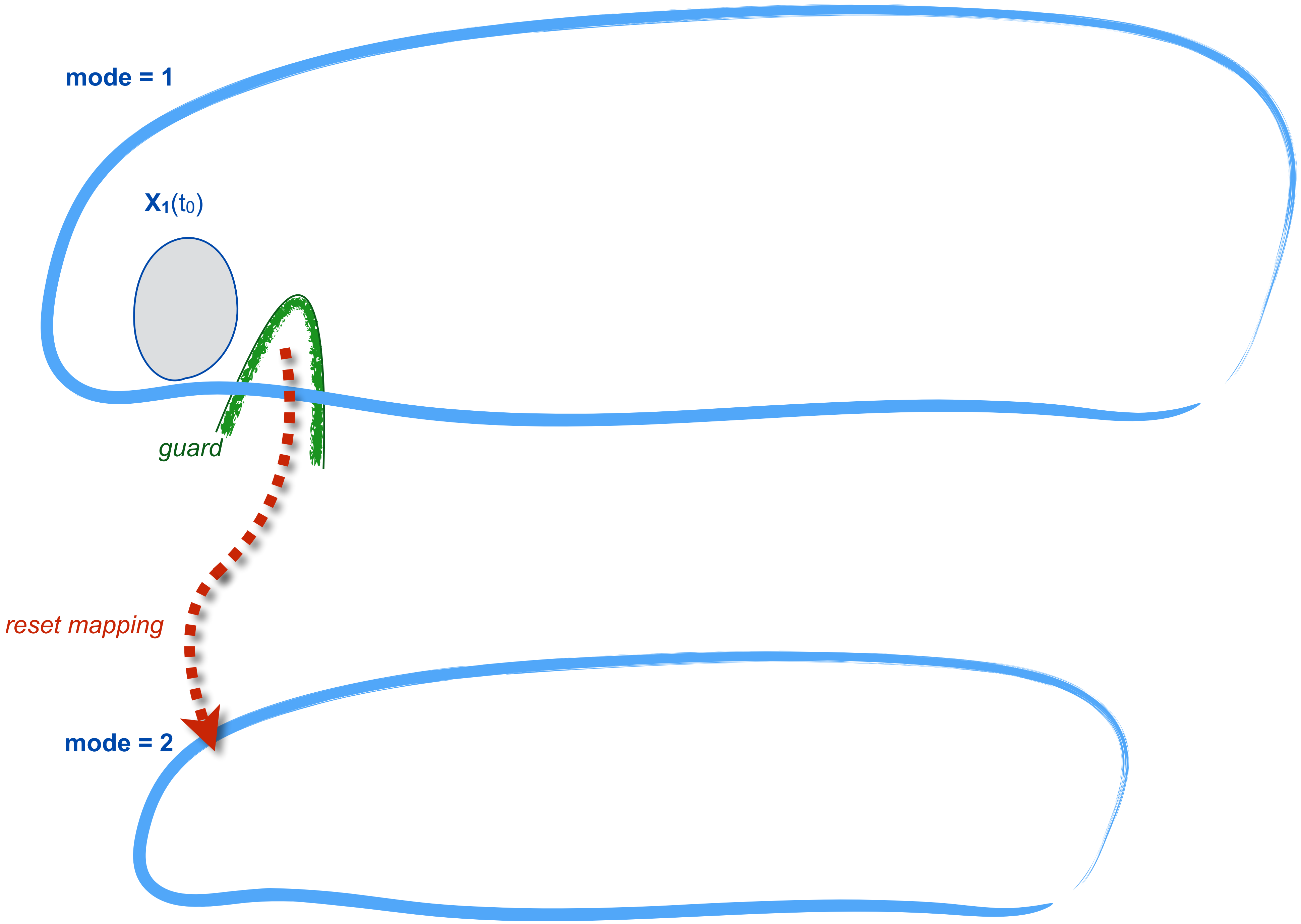

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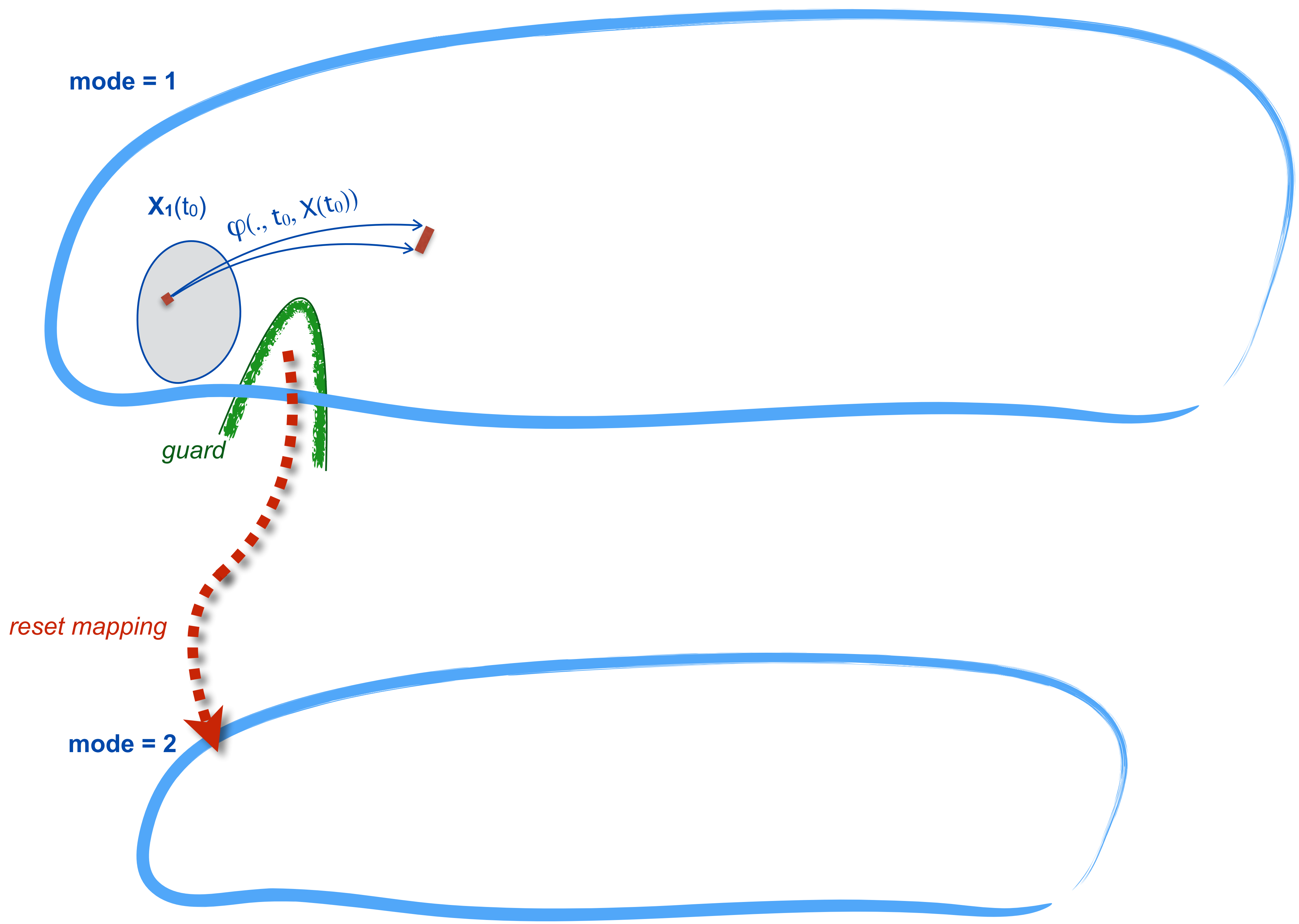

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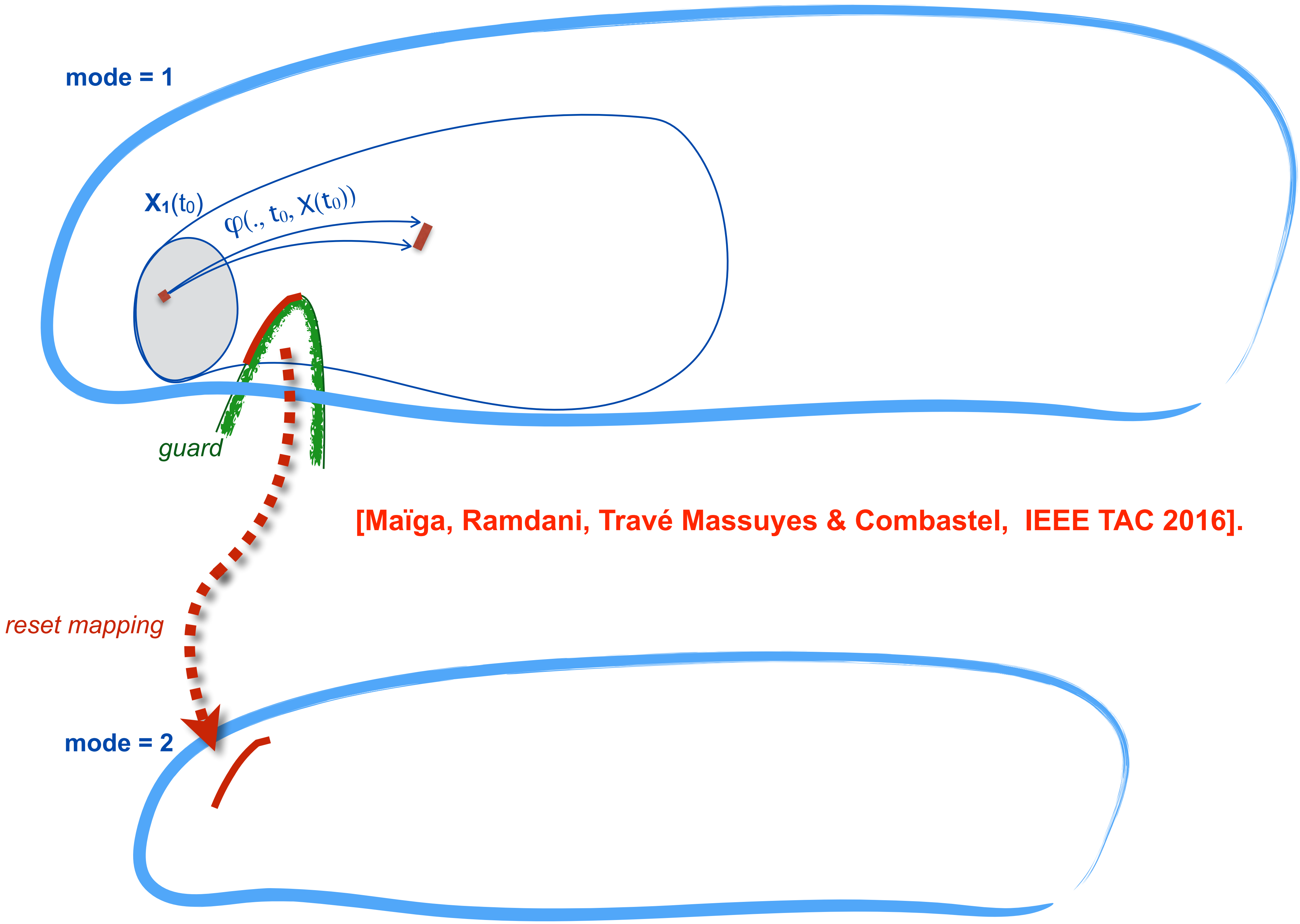
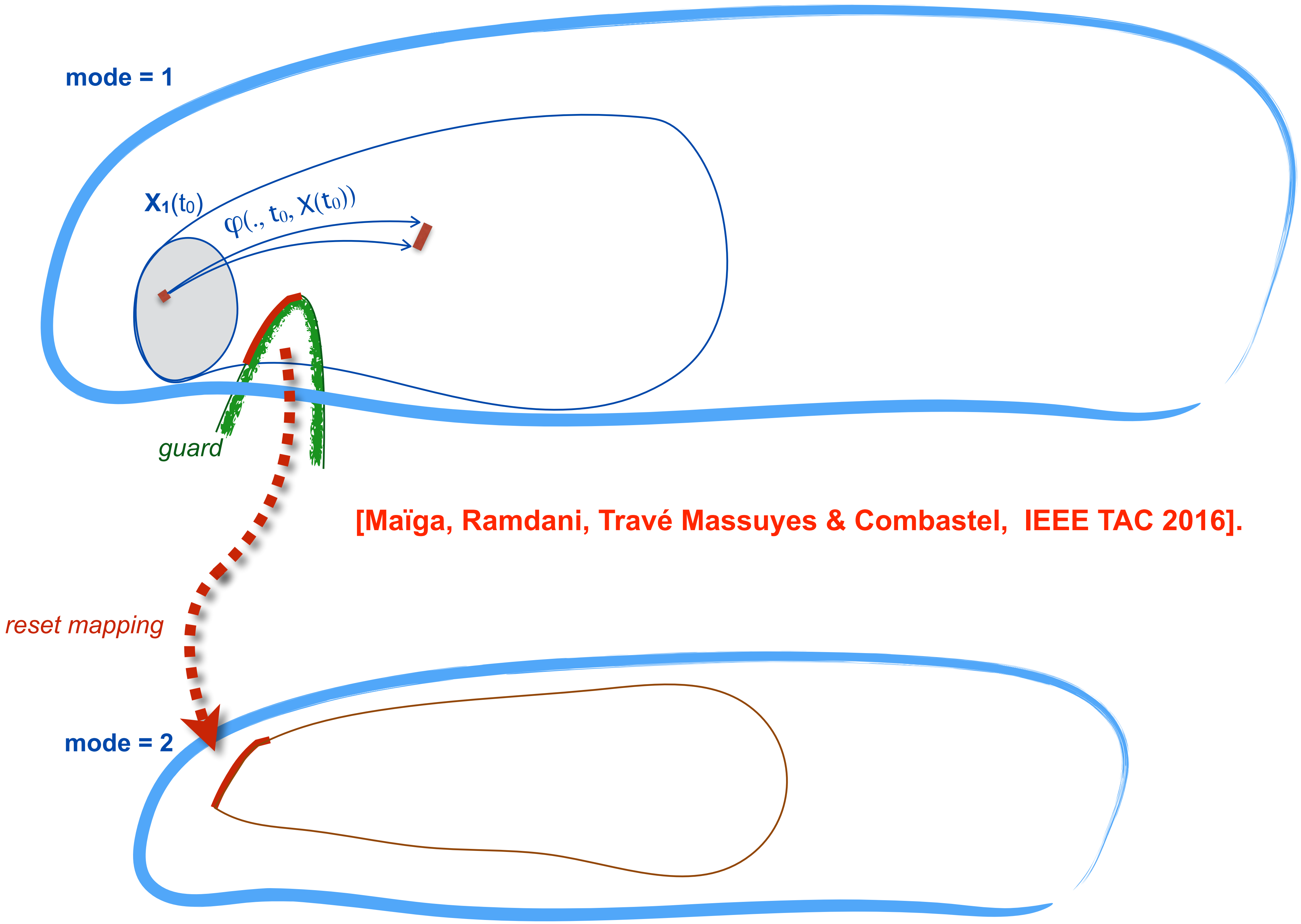


[Maïga, Ramdani, Travé Massuyes & Combastel, IEEE TAC 2016].

Reachability based approach



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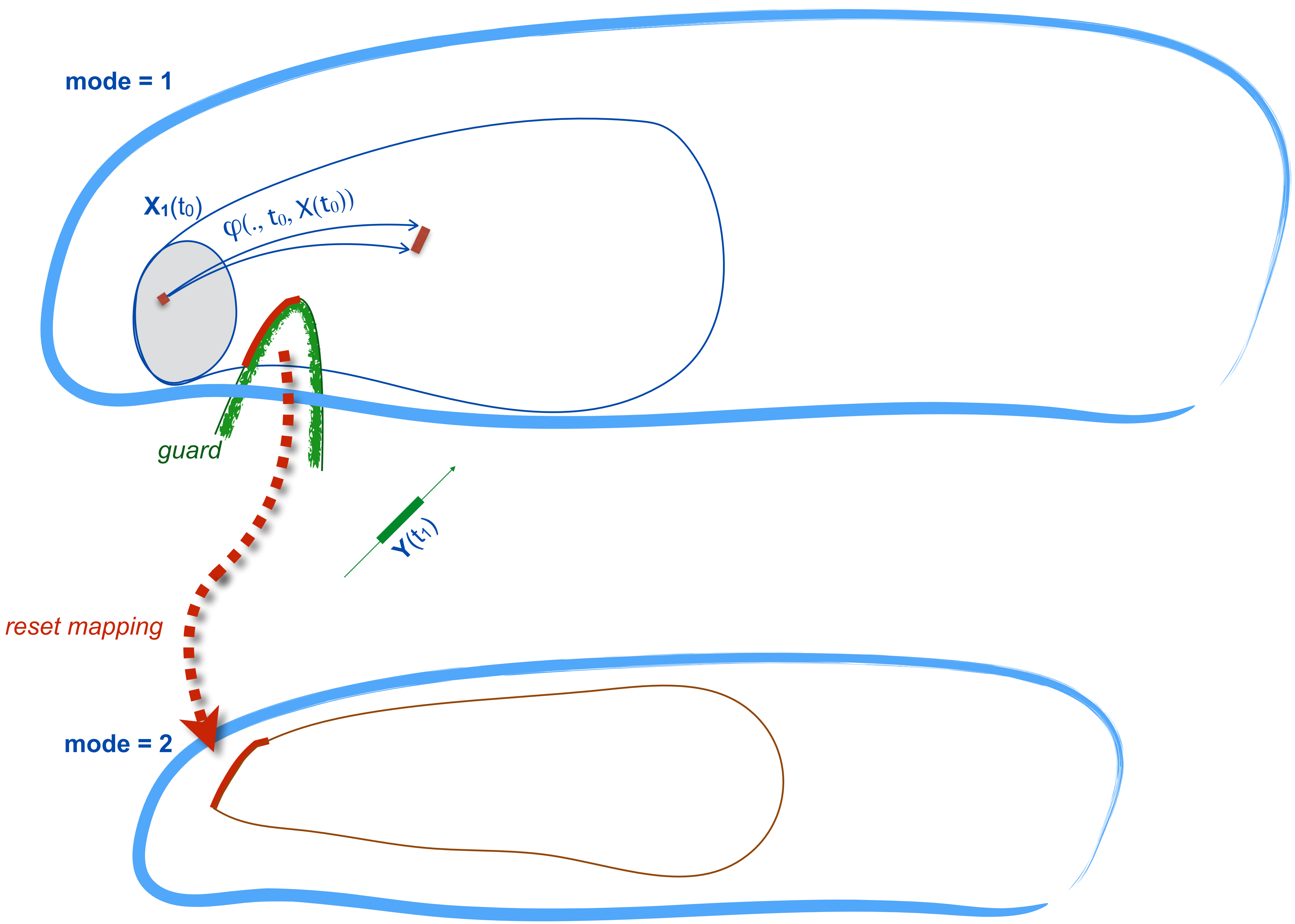

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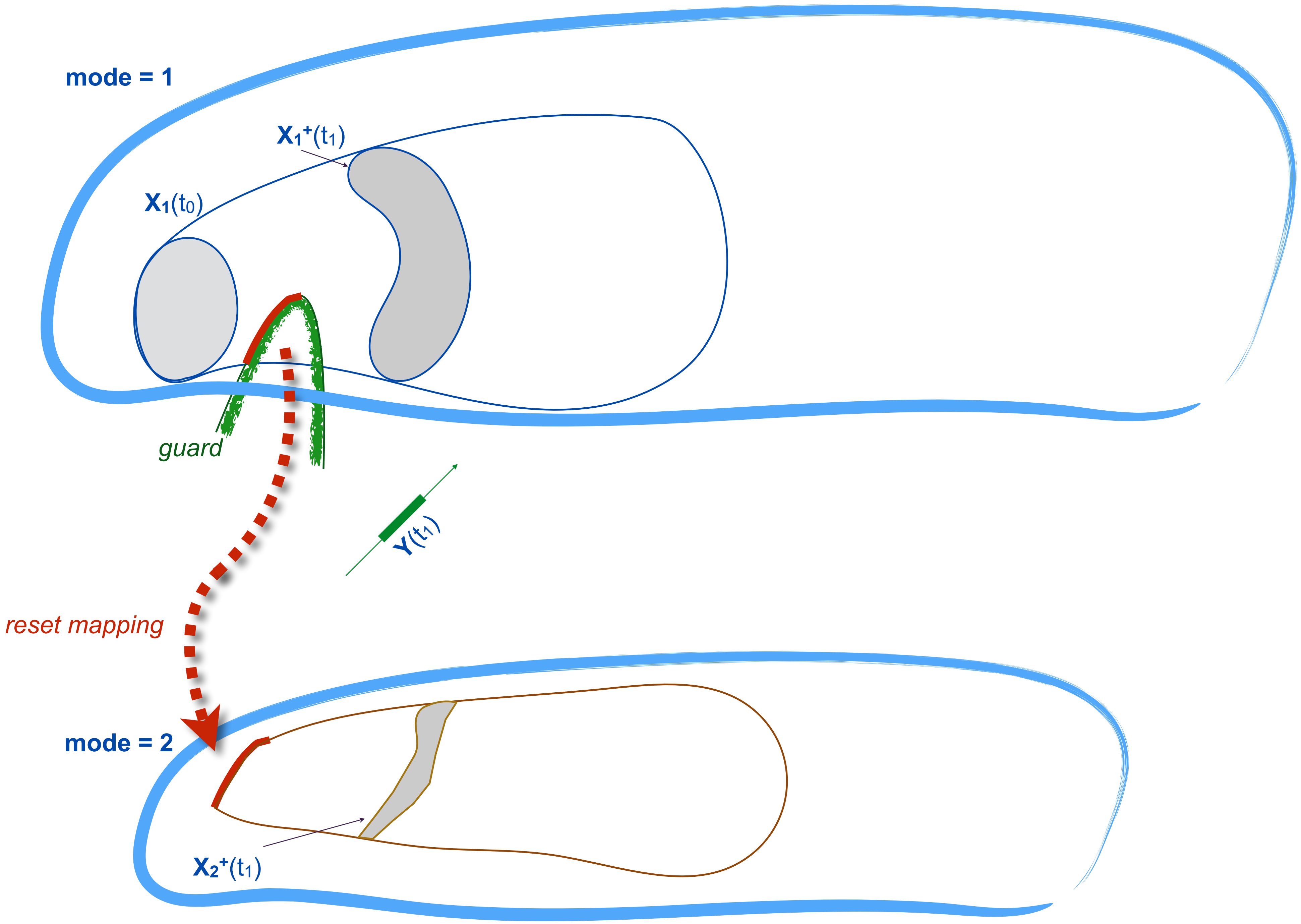

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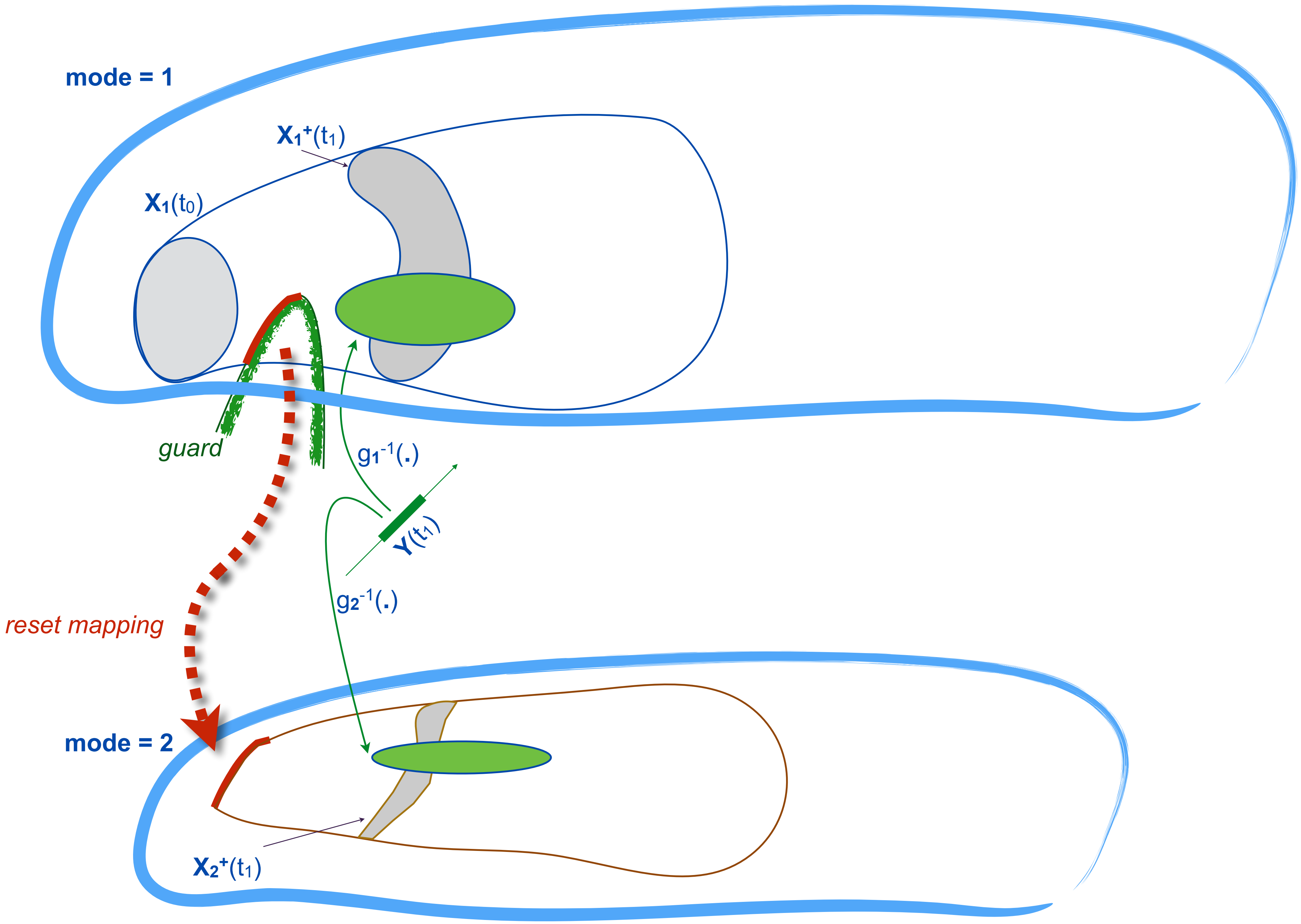

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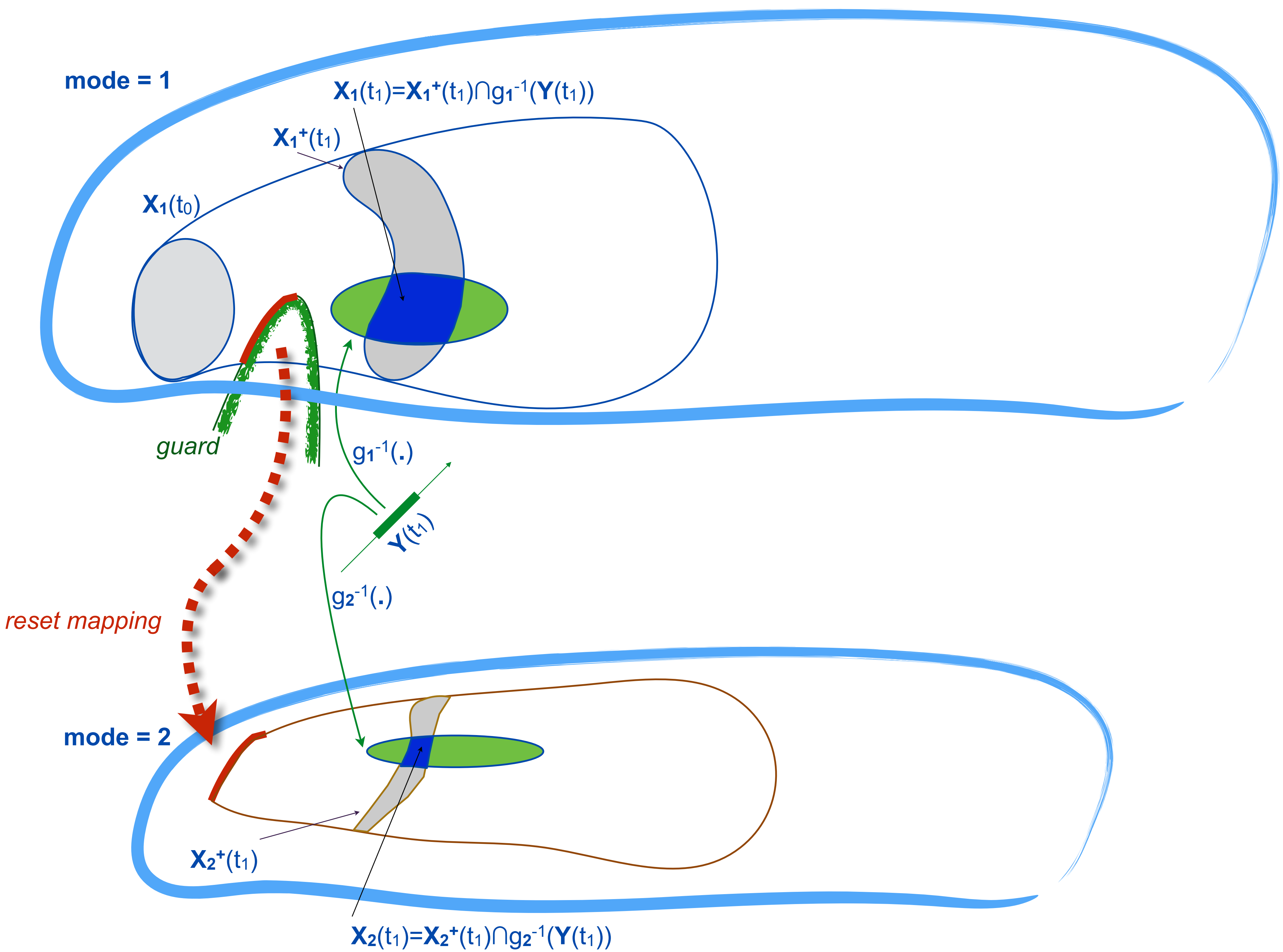

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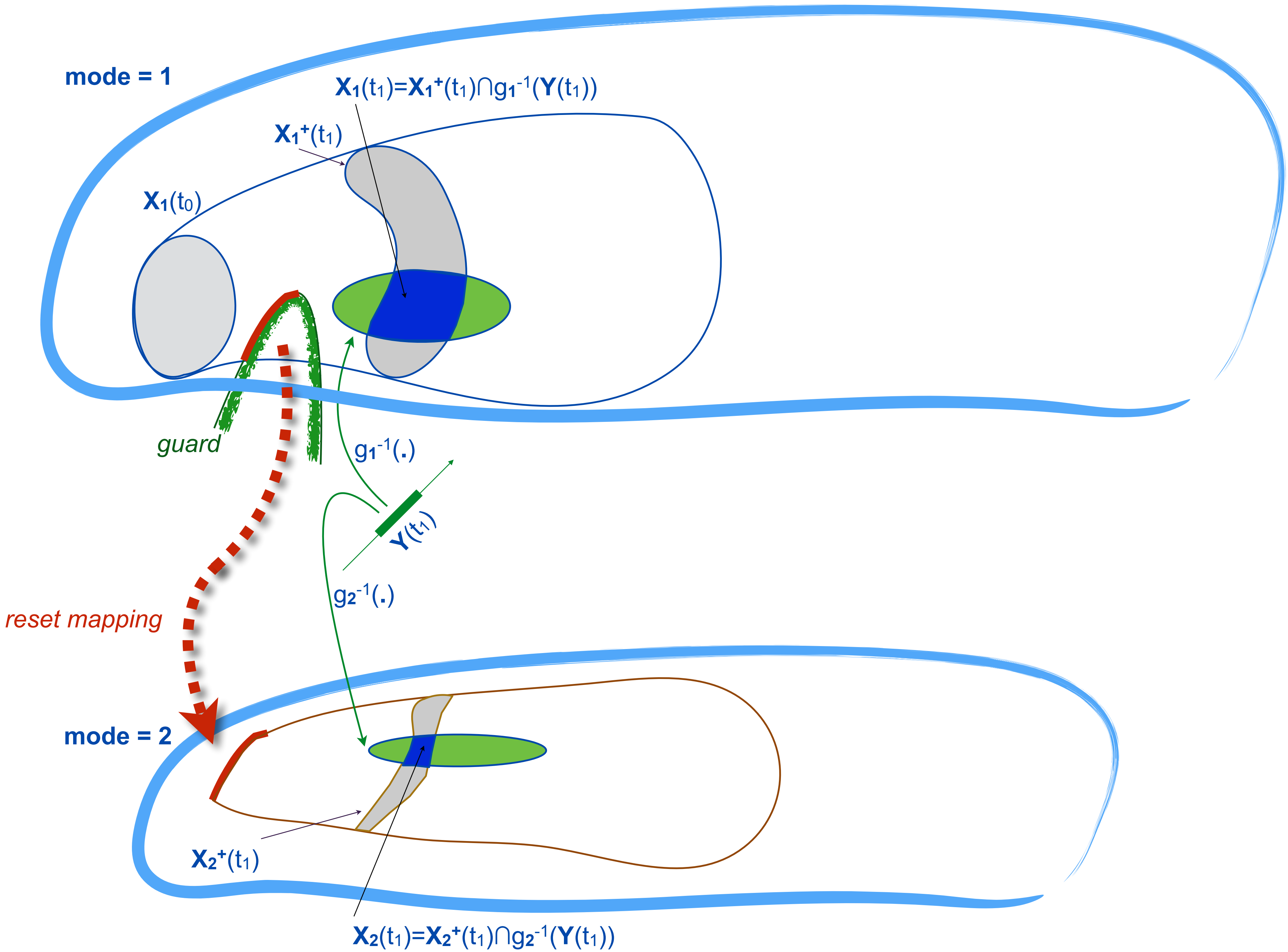


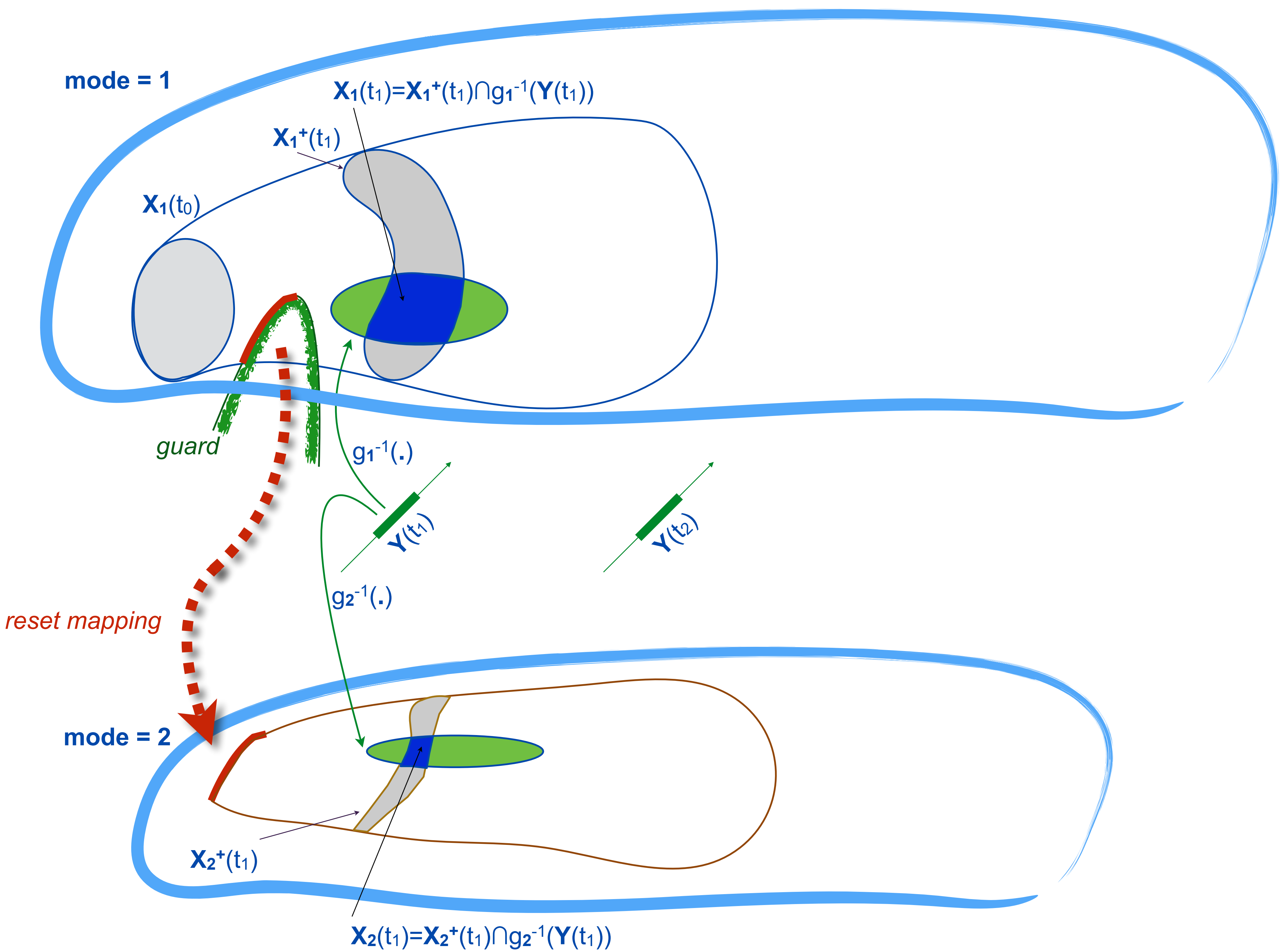
Reconstructed Hybrid Solution State Trajectory:

t_1

$\{q=1, \mathbf{X_1}(t_1)\} \cup$

$\{q=2, \mathbf{X_2}(t_1)\}$

Reachability based approach



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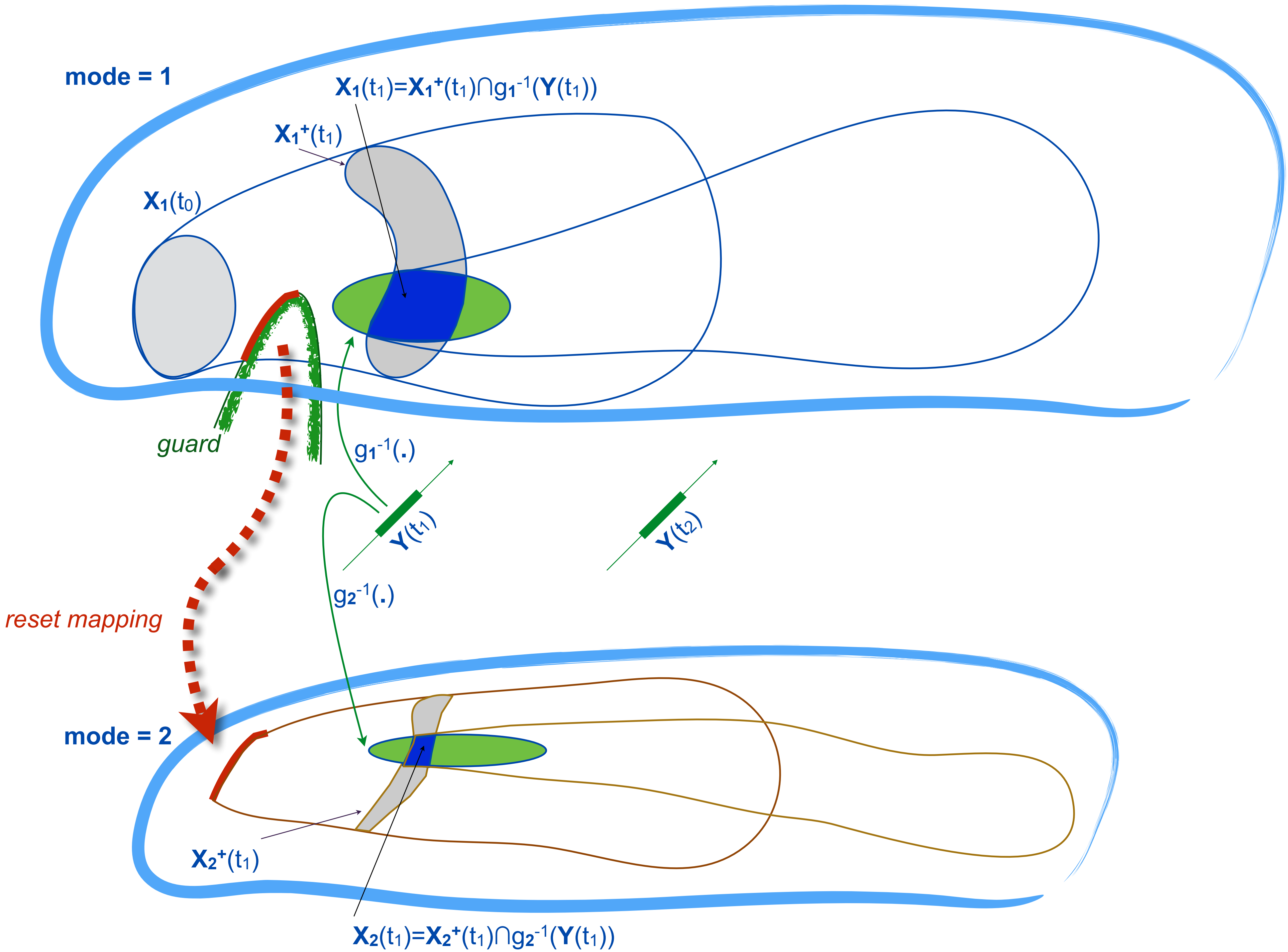

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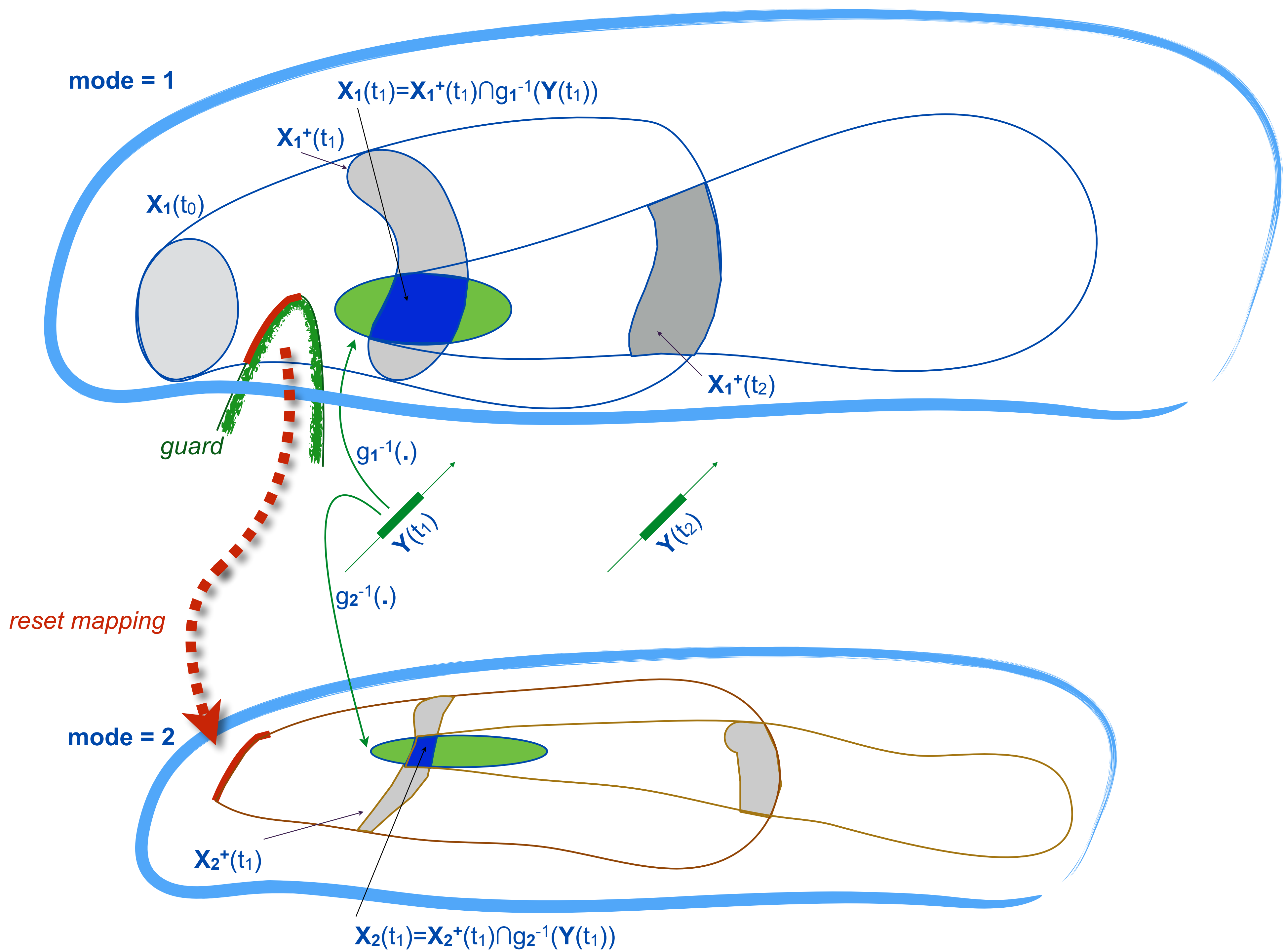

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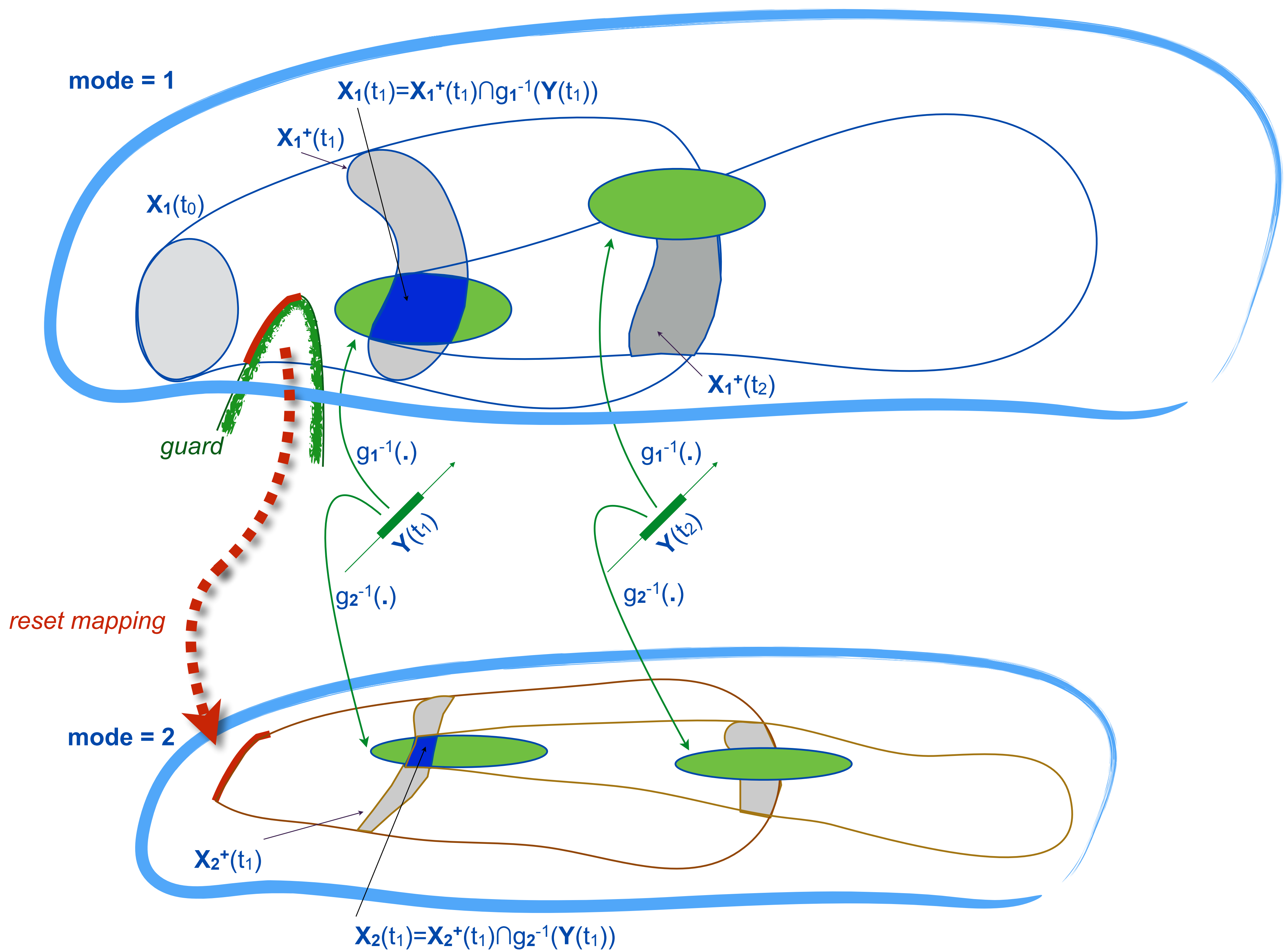

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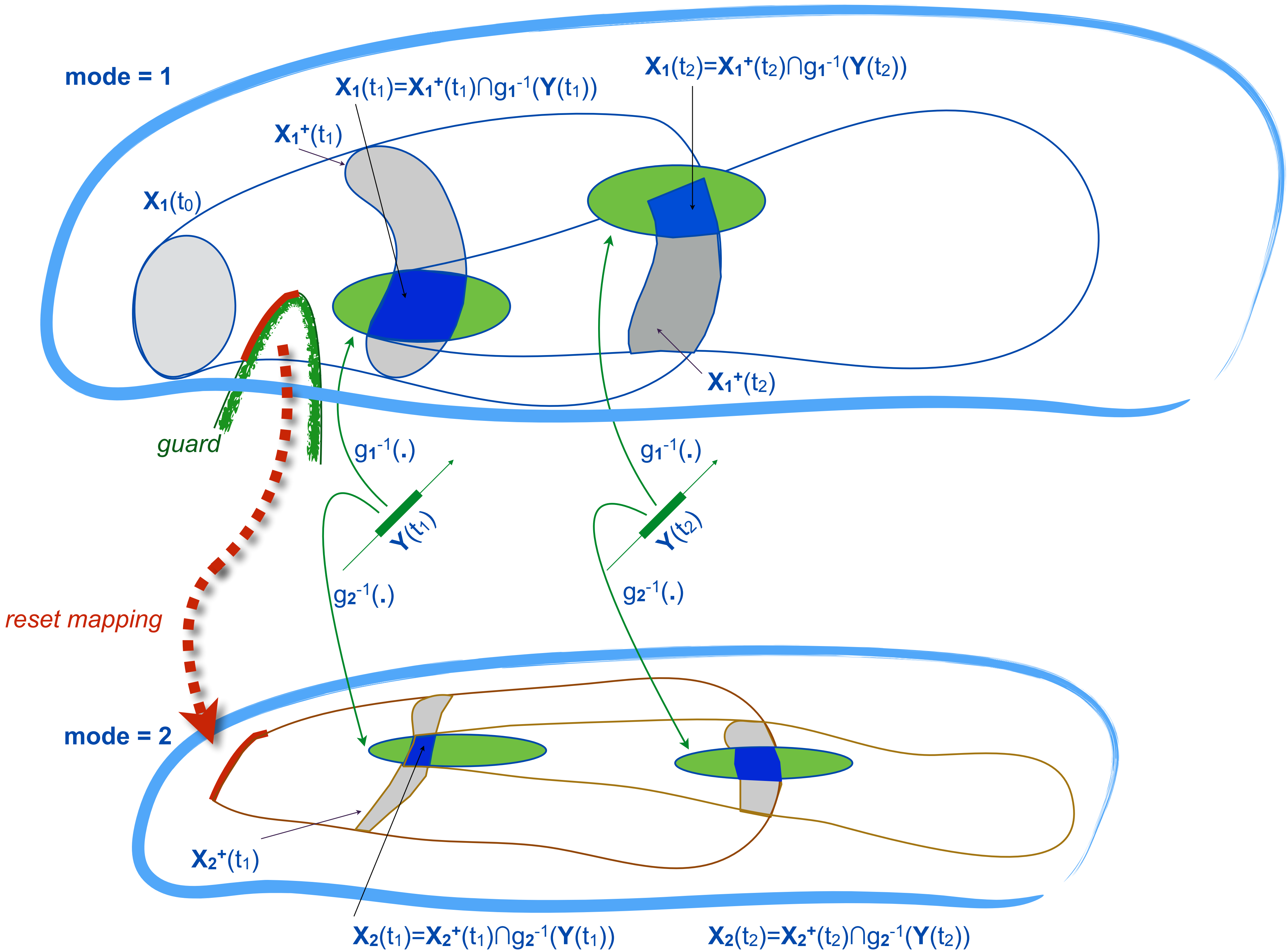

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t_2
$\{q=1, \mathbf{X_1}(t_2)\} \cup$
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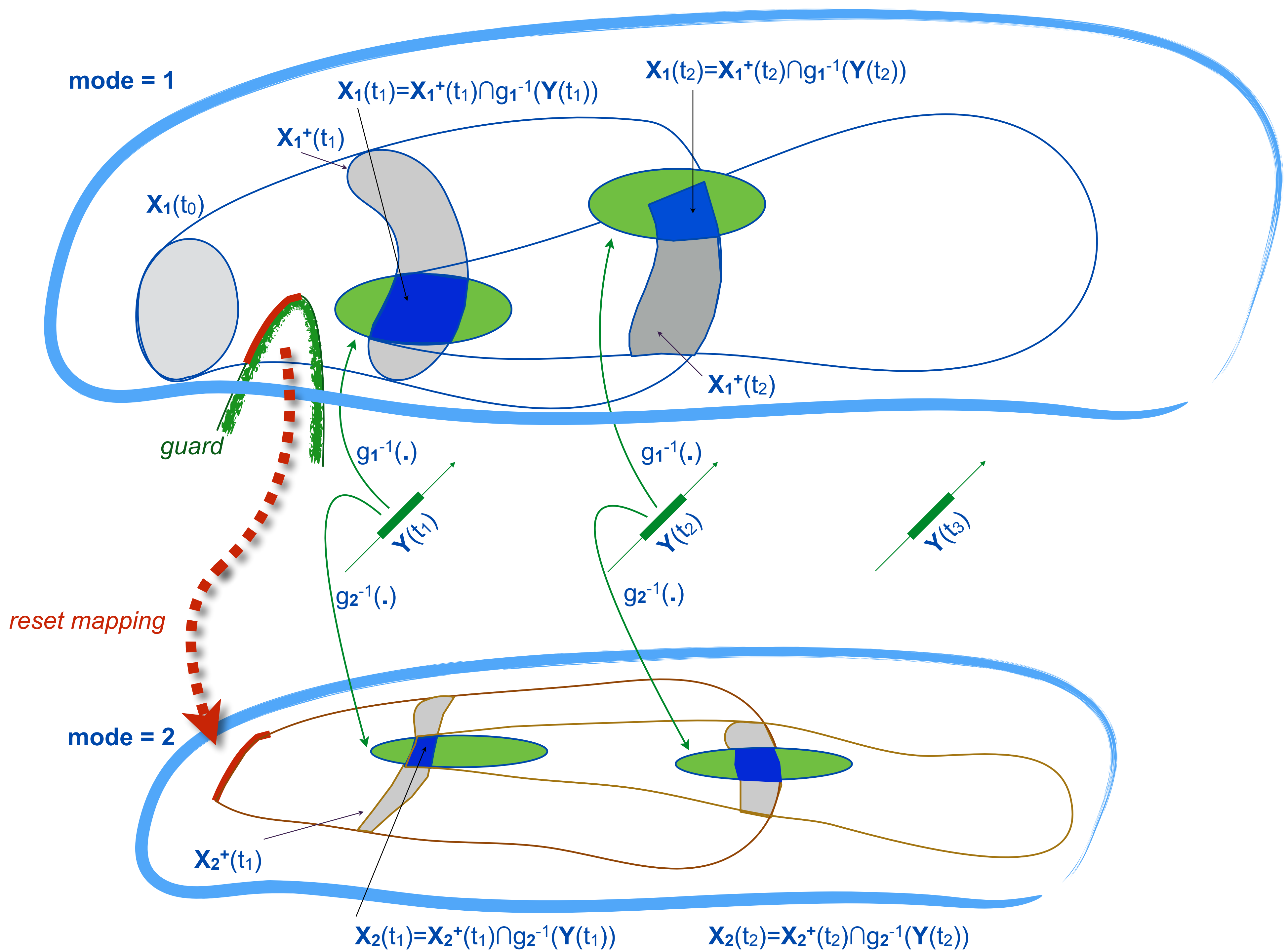
Reachability based approach



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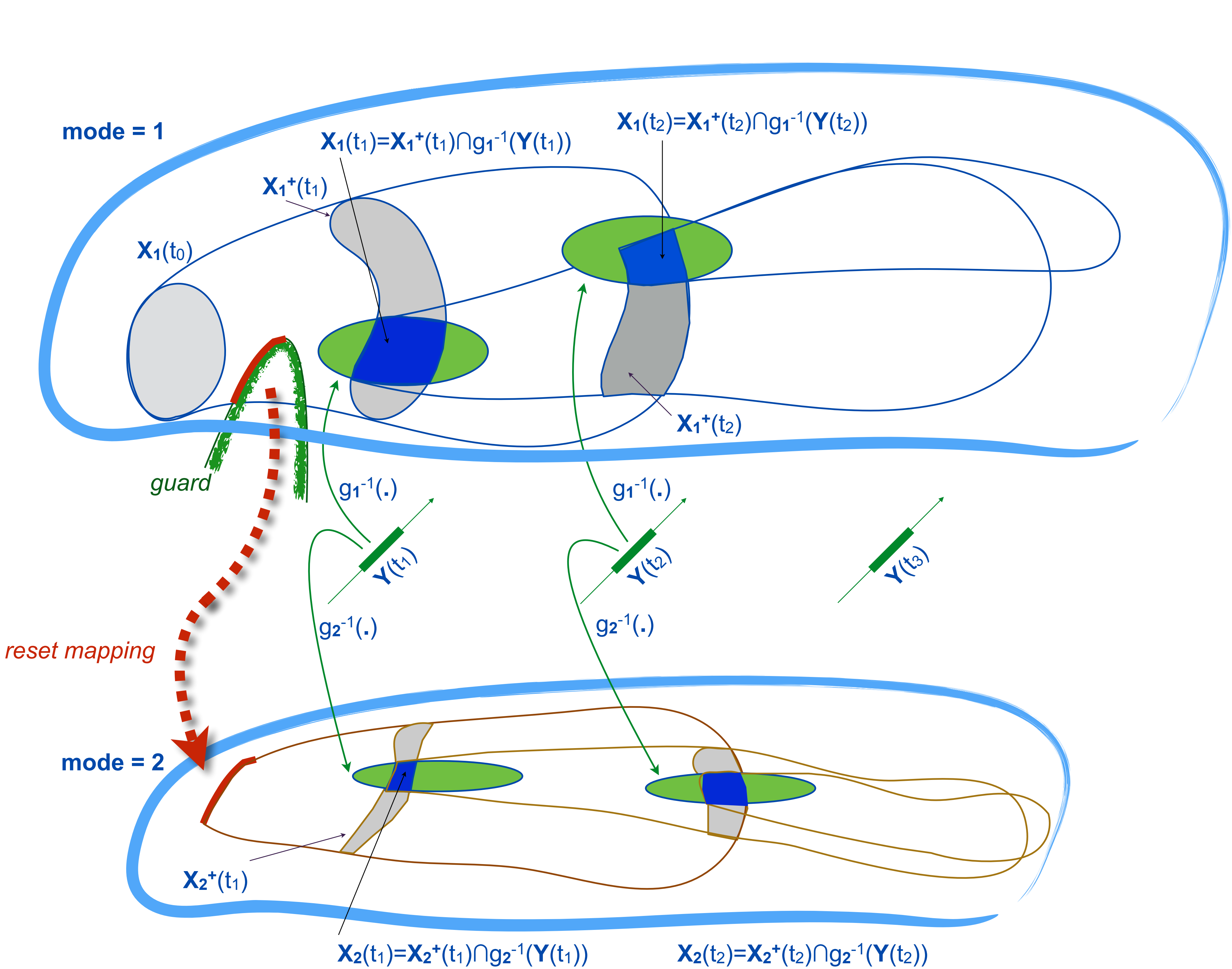

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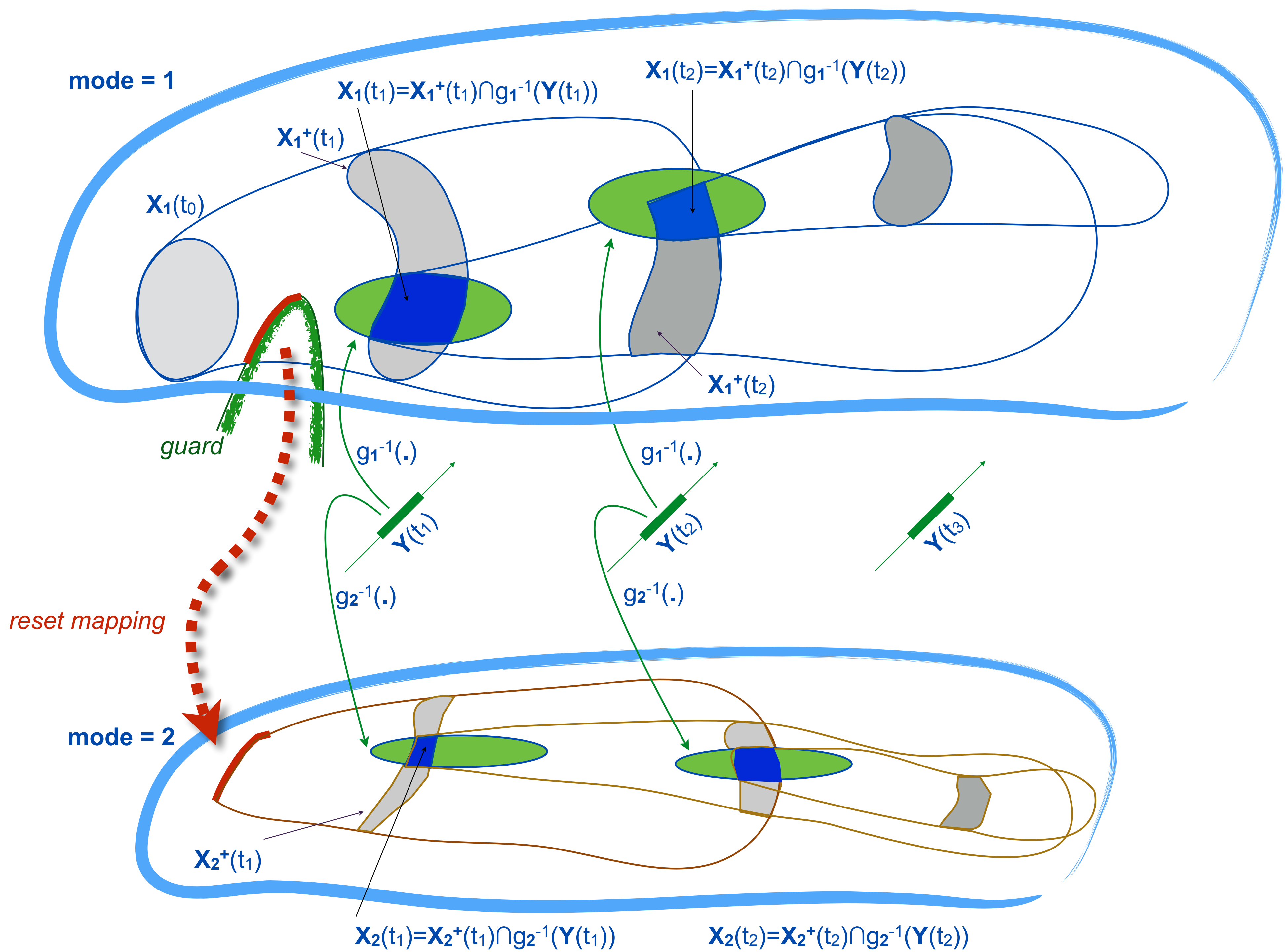

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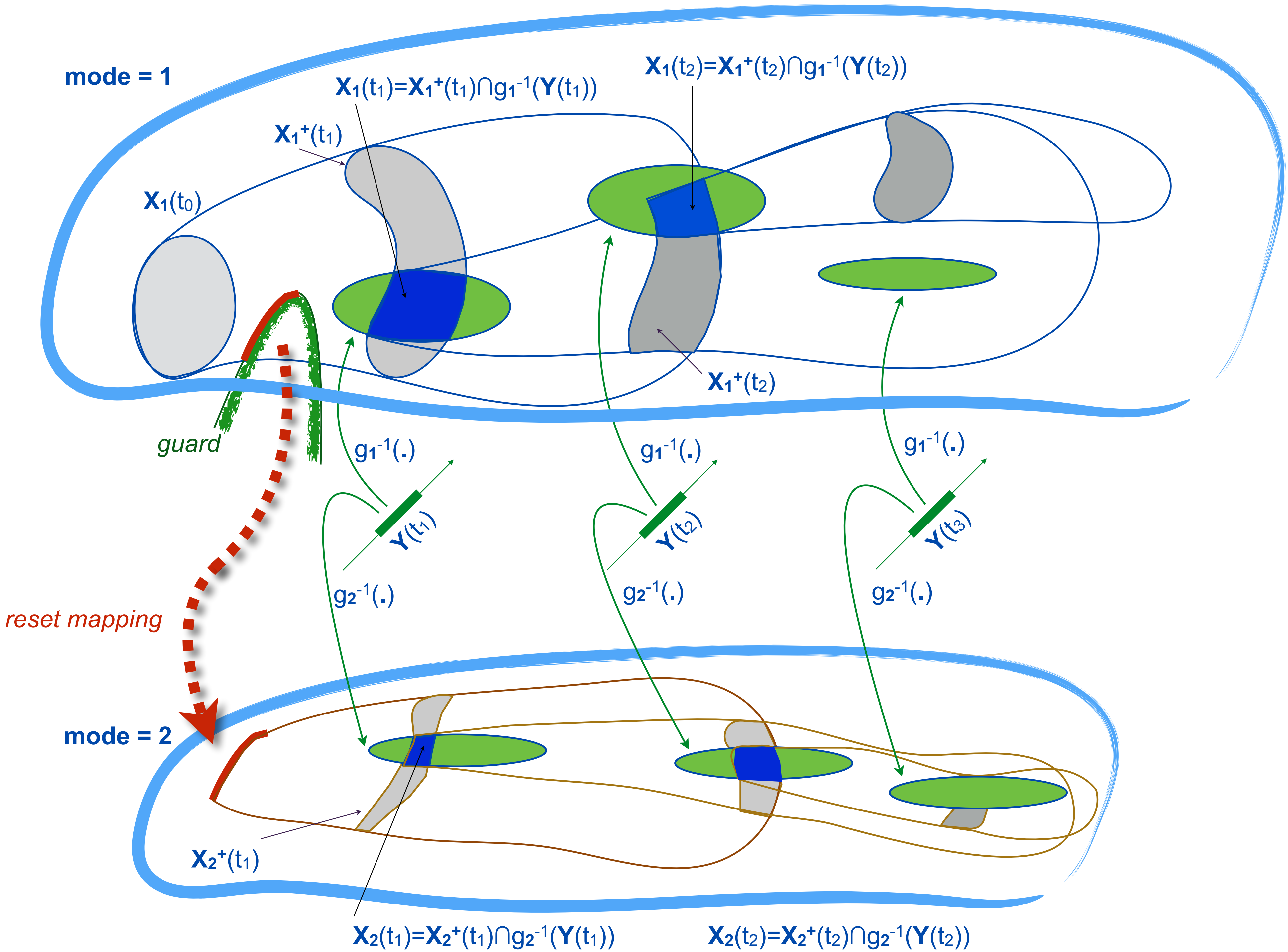

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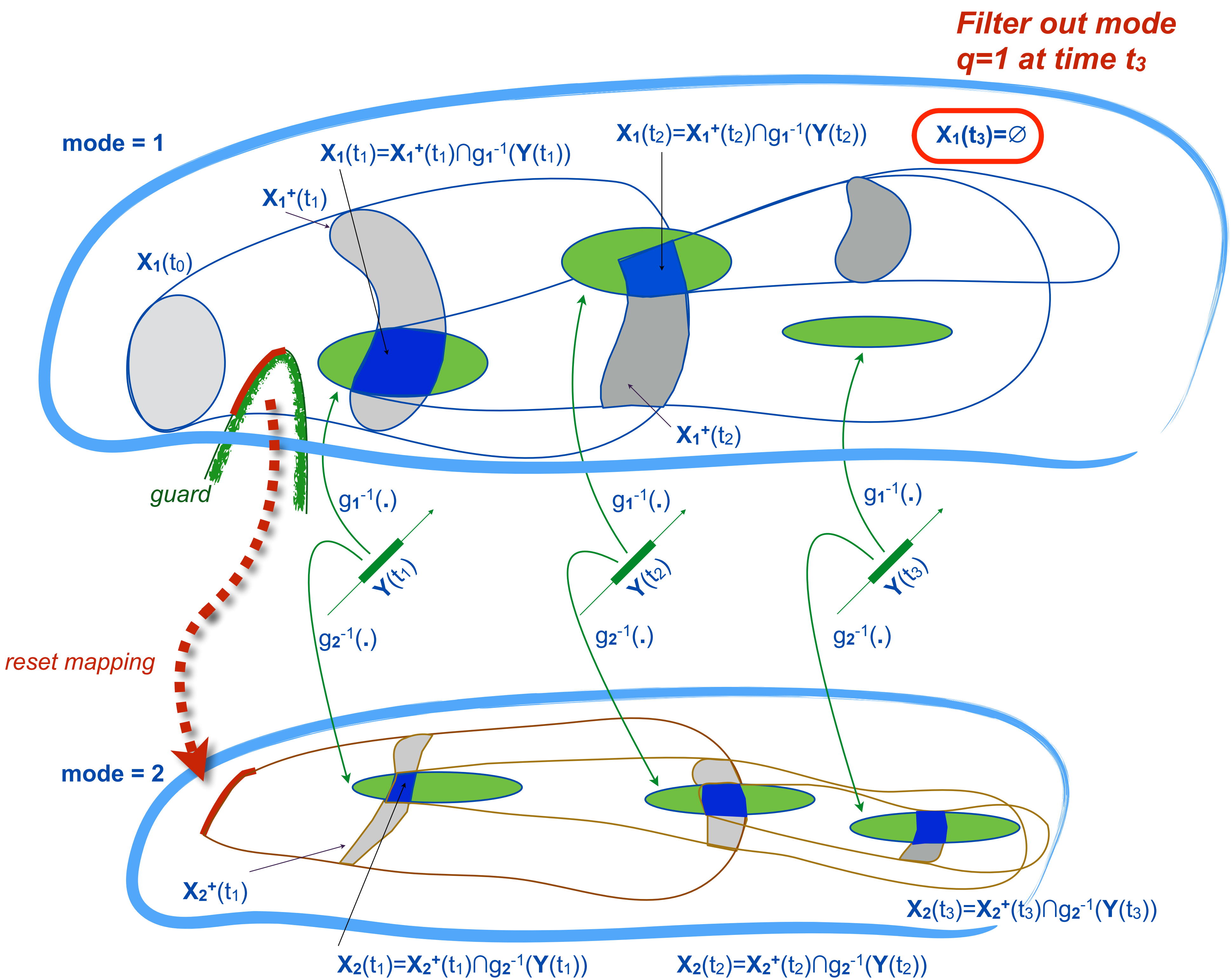

Reconstructed Hybrid Solution State Trajectory:

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t_2
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$\{q=2, \mathbf{X}_2(t_2)\}$

t_3
$\{q=2, \mathbf{X}_2(t_3)\}$

t_4

…

Continuous Reachability

Reachable set

$$\mathbb{R}([t_0,\ t]\,;\mathbb{X}_0) = \left\{ \begin{array}{l} \mathbf{x}(\tau),\ \ t_0 \le \tau \le t \mid \\ \dot{\mathbf{x}}(\tau) = f(\mathbf{x}, \mathbf{p}, \tau)\ \wedge\ \mathbf{x}(t_0) \in \mathbb{X}_0\ \wedge\ \mathbf{p} \in \mathbb{P} \end{array} \right\}$$

- Set integration
  - Interval Taylor methods
  - Bracketing enclosures

Continuous Reachability

Reachable set

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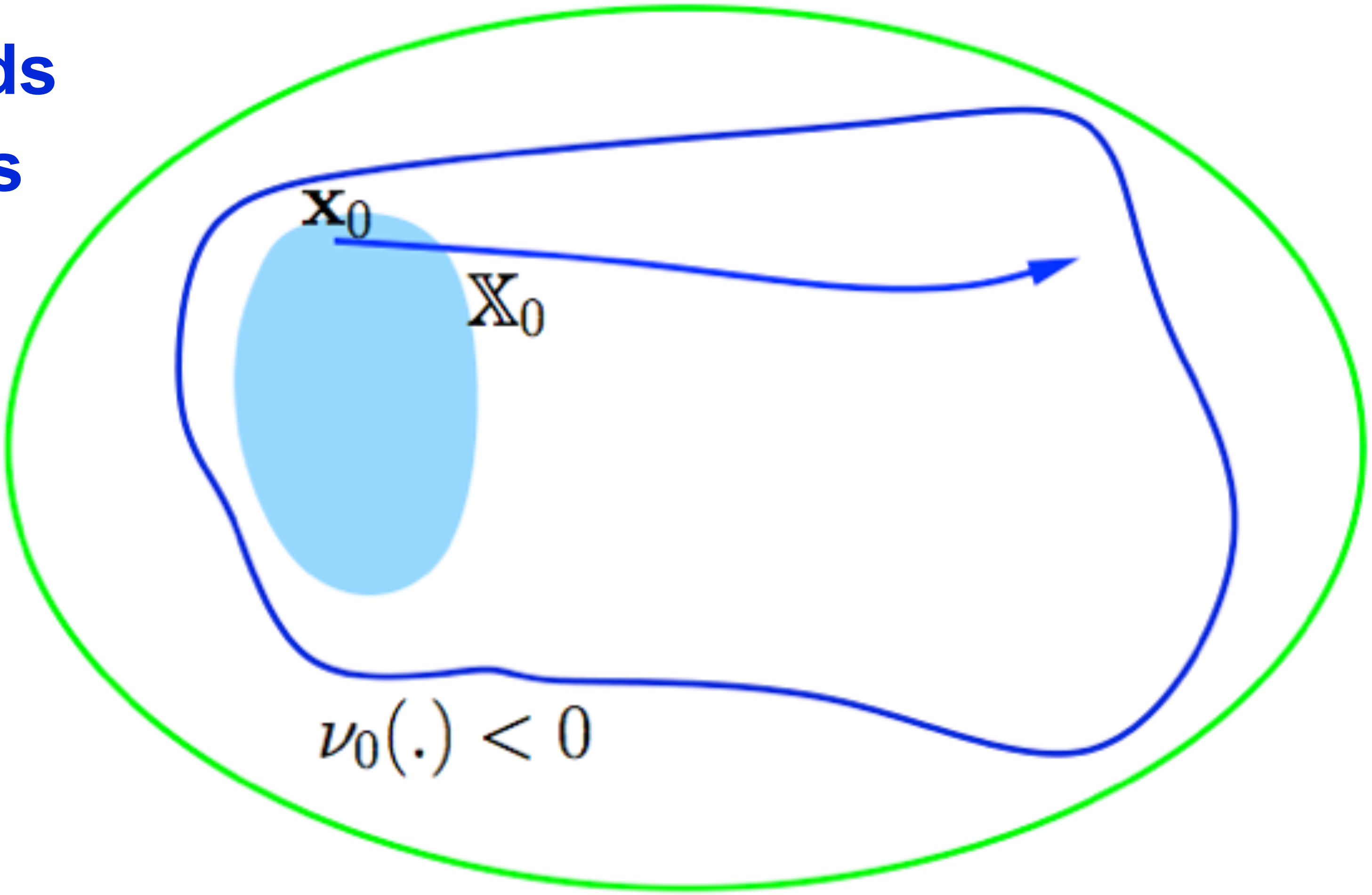
- **Set integration**
  - **Interval Taylor methods**
  - **Bracketing enclosures**



Continuous Reachability

Continuous Reachability

Guaranteed set integration

- **… with interval Taylor methods.**
  - (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)
- **… with interval Taylor models.**
  - (Chen, 2012)
- also via **interval Runge Kutta**.
  - (Alexandre dit Sandretto & Chapoutot, 2015)

Continuous Reachability

- **Guaranteed set integration**
  - **… with interval Taylor methods.**
    - (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)
  - **… with interval Taylor models.**
    - (Chen, 2012)
  - also via **interval Runge Kutta**.
    - (Alexandre dit Sandretto & Chapoutot, 2015)
- **Comparison theorems for differential inequalities**
  - **Monotone systems**
    - (Ramdani et al., 2010)
  - **Muller's theorem**
    - (Kieffer et al. 2006) (Ramdani, et al. 2006), (Ramdani, et al. 2009)

Hybrid Reachability Computation

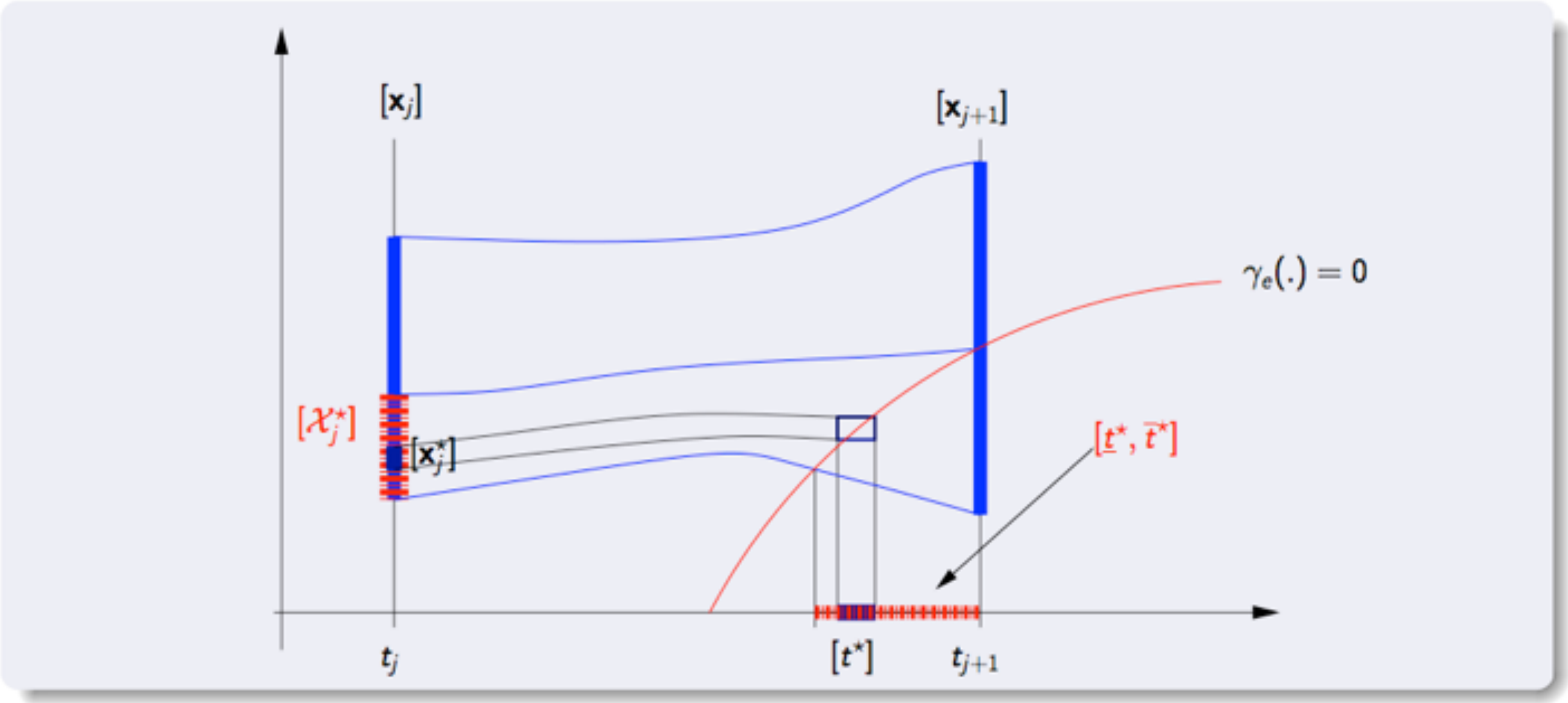
- **Guaranteed event detection & localization**
  - **An interval constraint propagation approach**
    - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

Hybrid Reachability Computation

Guaranteed event detection & localization

- **An interval constraint propagation approach**
  - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

Time grid $\rightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$



Compute $[\underline{t}^\star, \overline{t}^\star] \times [\mathcal{X}_j^\star]$

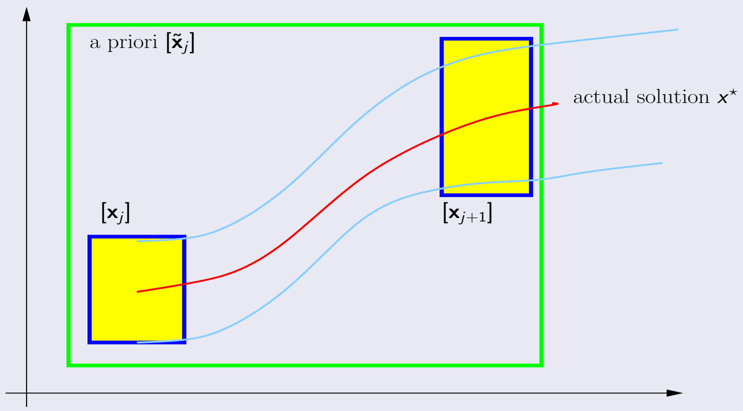
Hybrid Reachability Computation

Guaranteed event detection & localization

- **An interval constraint propagation approach**
  - (Ramdani & Nedialkov, Nonlinear Analysis Hybrid Systems 2011)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{p}, t), \quad t_0 \le t \le t_N,\ \mathbf{x}(t_0) \in [\mathbf{x}_0]\,,\ \mathbf{p} \in [\mathbf{p}]$$

Time grid $\longrightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$



- Analytical solution for $[\mathbf{x}](t),\ t \in [t_j, t_{j+1}]$

$$[\mathbf{x}](t) = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t - t_j)^k \mathbf{f}^{[k]}([\tilde{\mathbf{x}}_j], [\mathbf{p}])$$

Hybrid Reachability Computation

Guaranteed event detection & localization

- **An interval constraint propagation approach**
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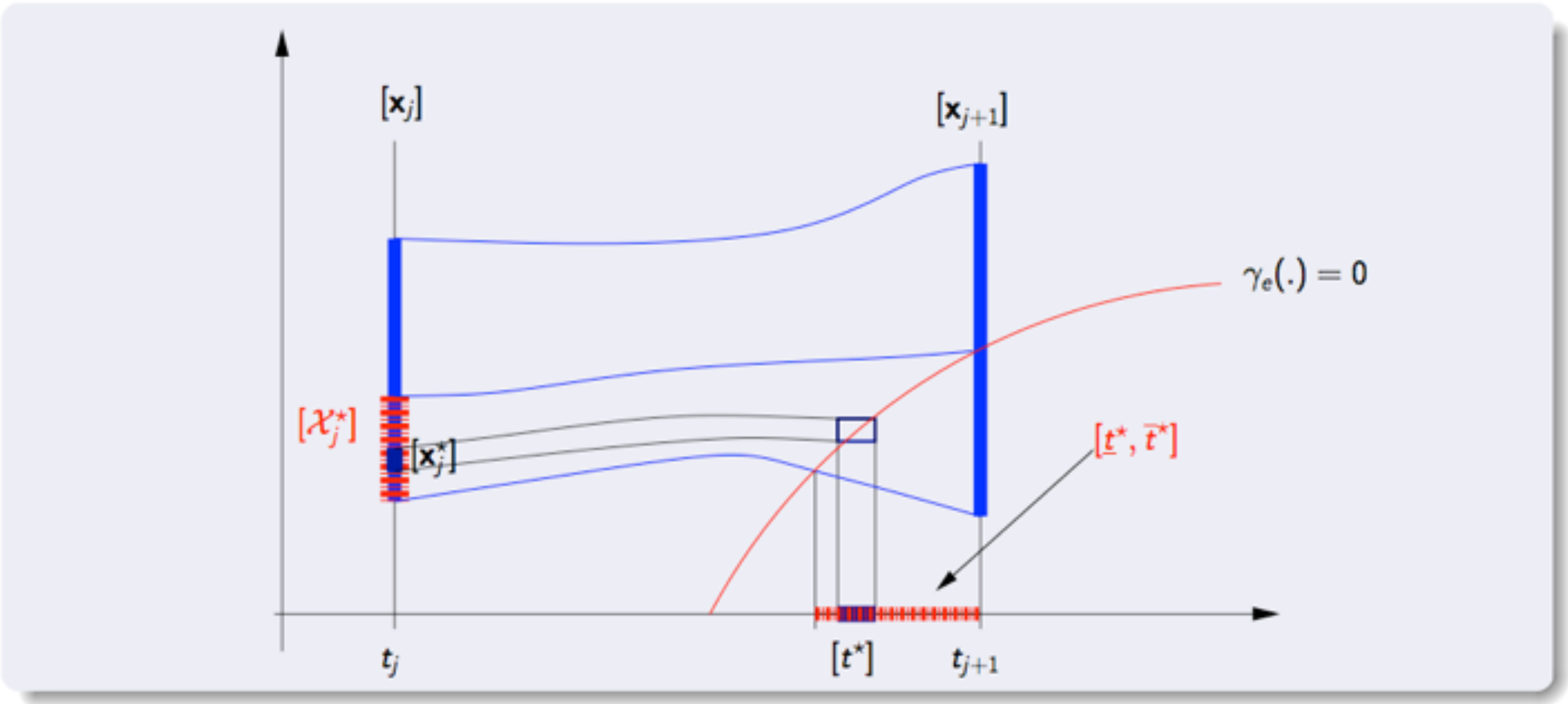
Time grid → $t_0 < t_1 < t_2 < \cdots < t_N$



Compute $[\underline{t}^\star, \overline{t}^\star] \times [\mathcal{X}_j^\star]$

Hybrid Reachability Computation

Guaranteed event detection & localization

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Time grid $\rightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N$

- $[\mathbf{x}](t) = \text{Interval Taylor Series (ITS)}(t, [\mathbf{x}_j], [\tilde{\mathbf{x}}_j])$
- $\gamma([\mathbf{x}](t)) = 0$

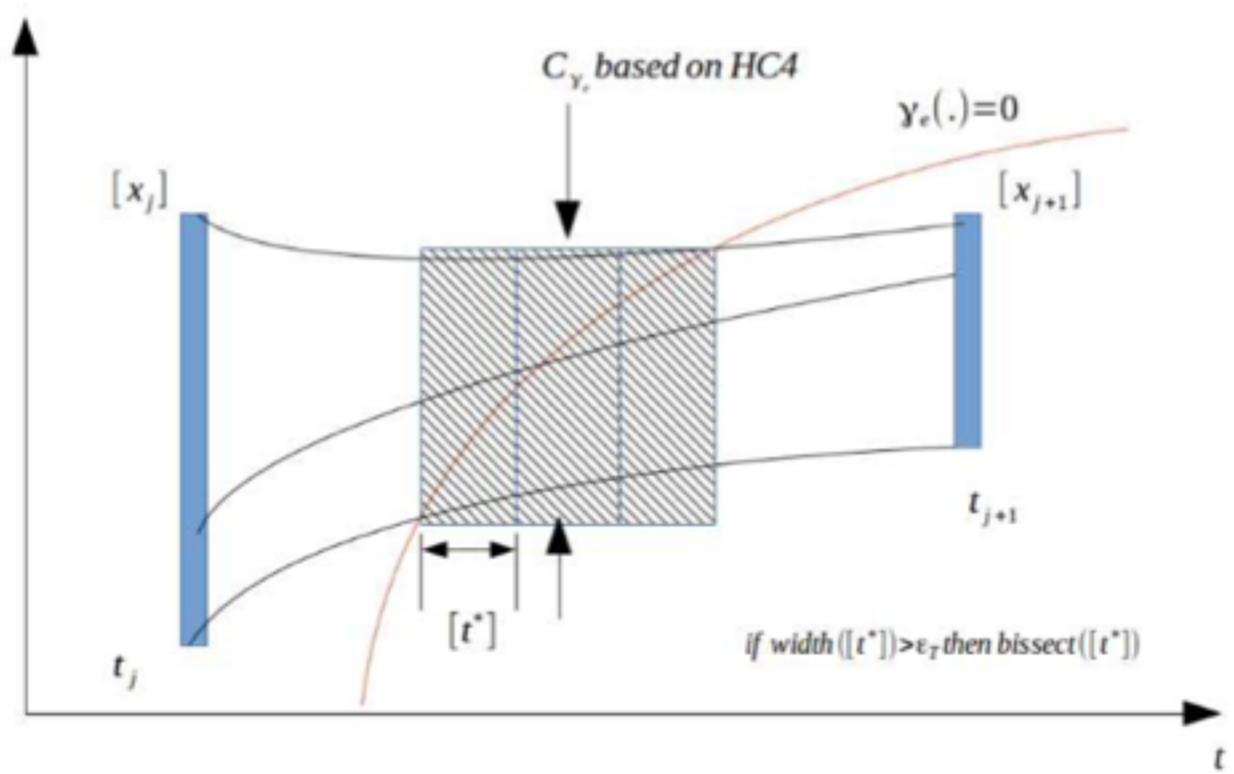
$\Rightarrow \gamma \circ \text{ITS}(t, \mathbf{x}_j, [\tilde{\mathbf{x}}_j]) \rightarrow \psi(t, \mathbf{x}_j)$

Solve CSP $([t_j, t_{j+1}] \times [\mathbf{x}_j], \psi(.,.) \ni 0)$

Hybrid Reachability Computation

Detecting and localizing events

- **Improved and enhanced version. A faster version.**
  - (Maïga, Ramdani, et al., IEEE CDC 2013, ECC 2014)



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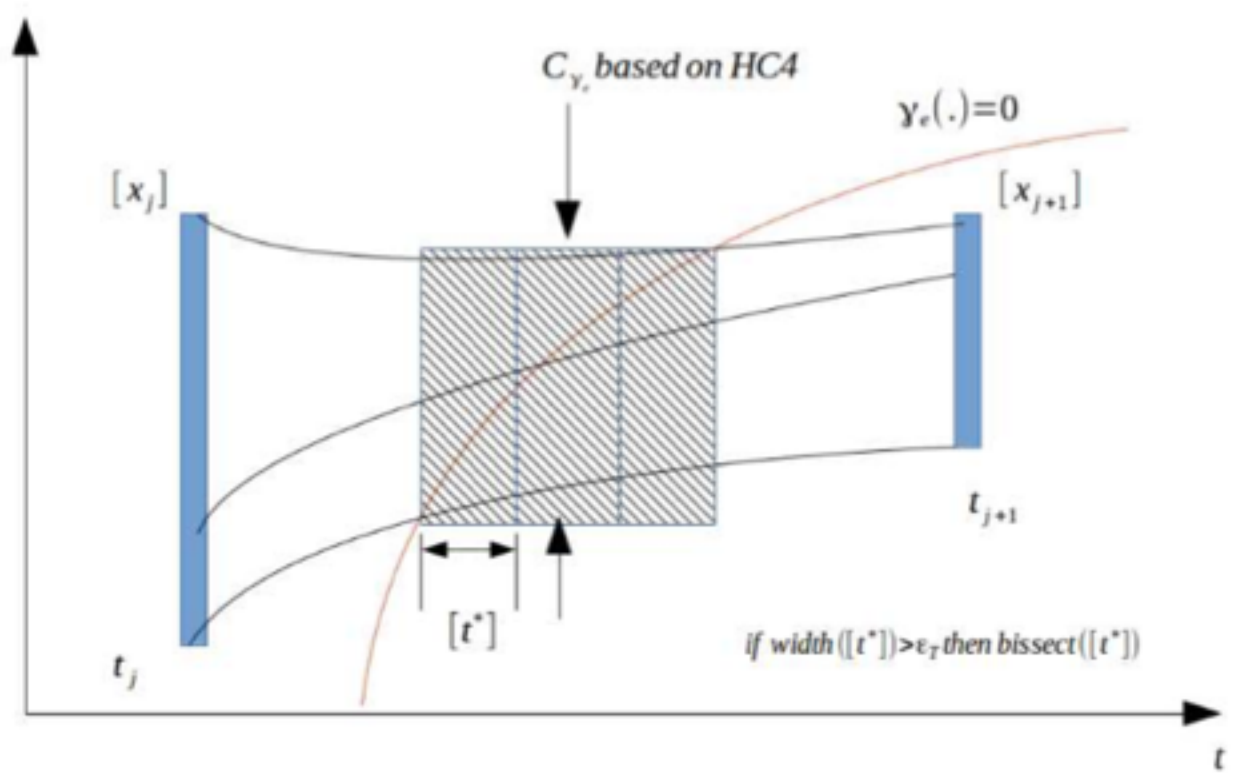
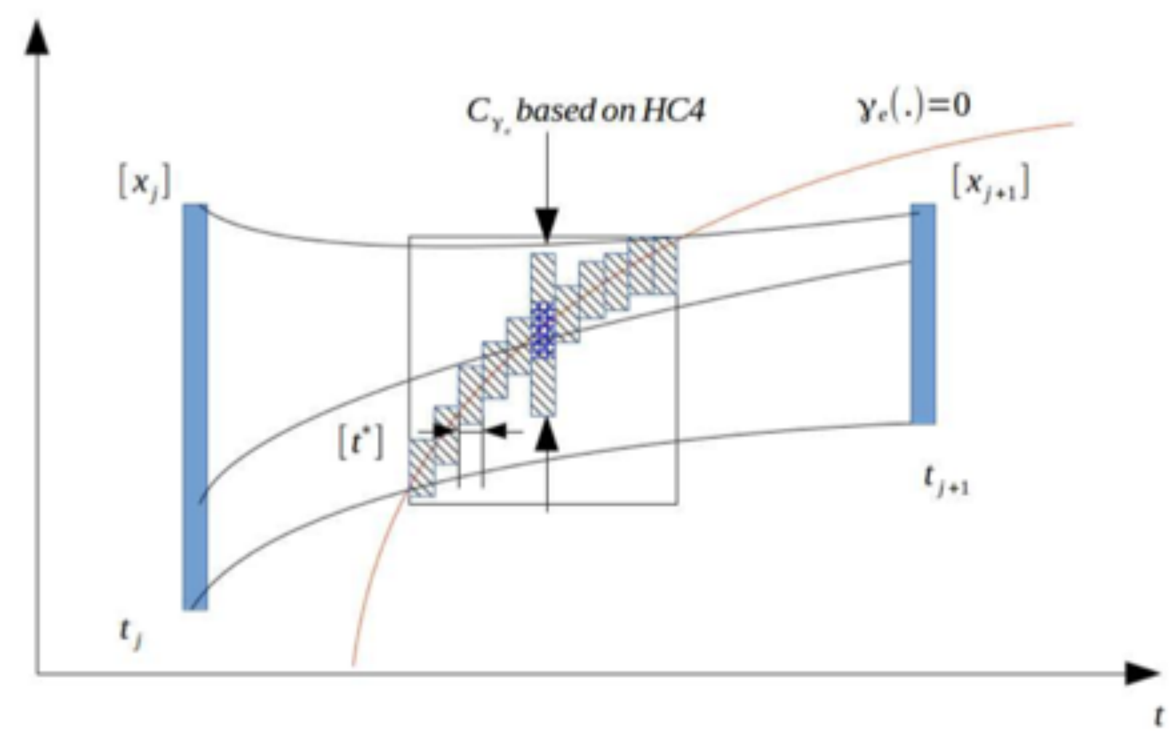
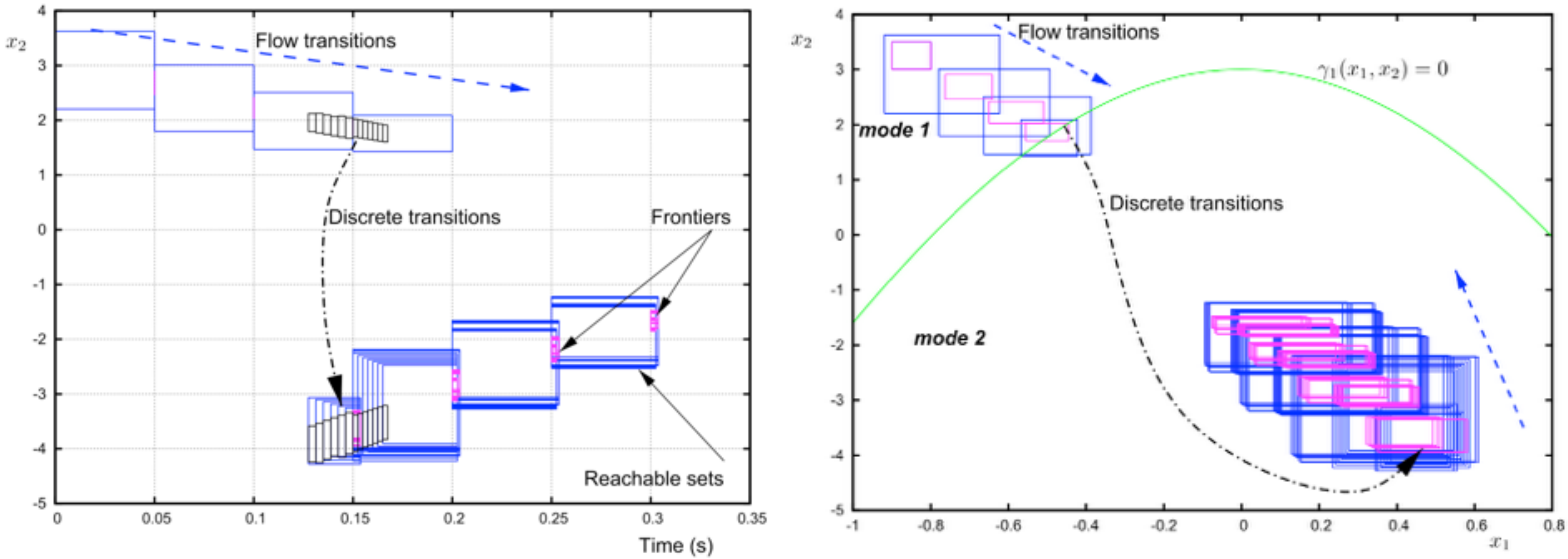




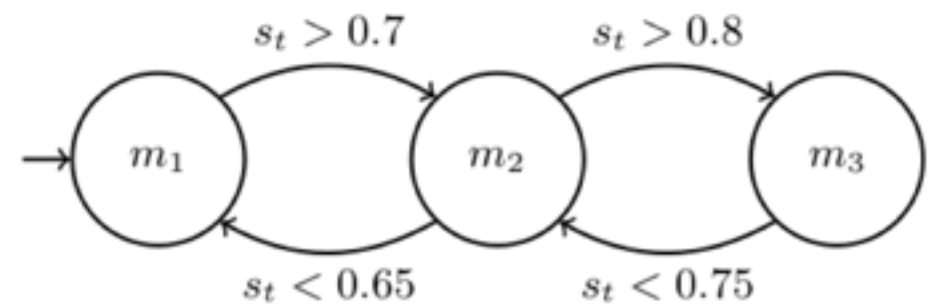
Hybrid Reachability Computation

Detecting and localizing events

- **Improved and enhanced version**
  - (Maïga, Ramdani, et al., IEEE CDC 2013, ECC 2014)



Hybrid Reachability Computation



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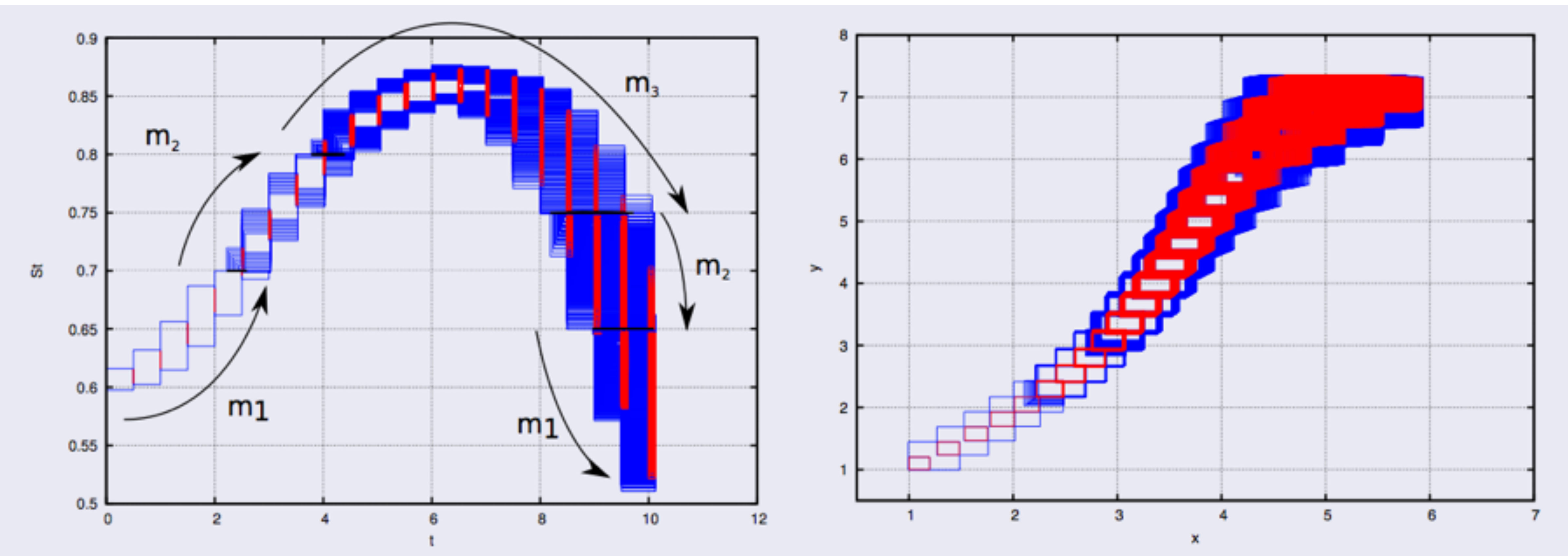

(e) St×t

(f) Y×X space

Hybrid Reachability Computation

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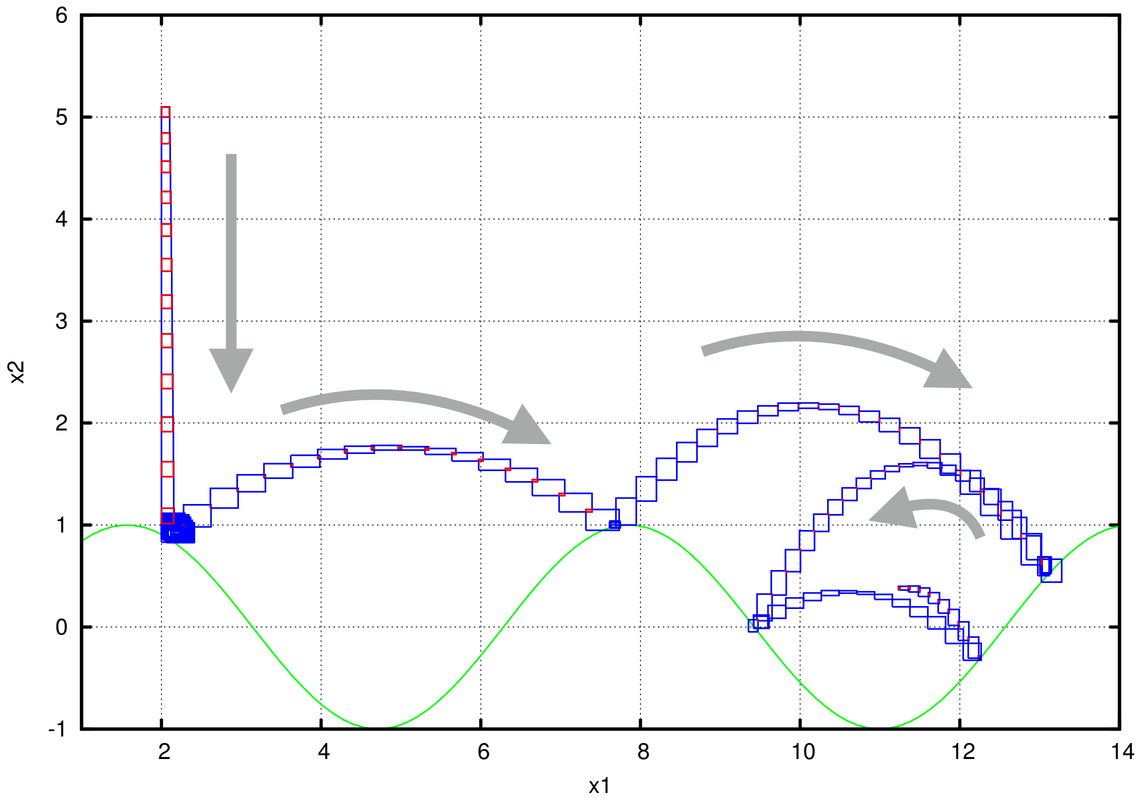
Bouncing ball in 2D.



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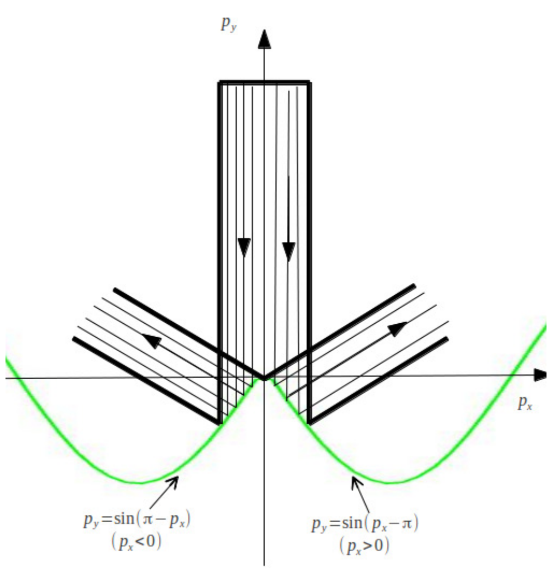
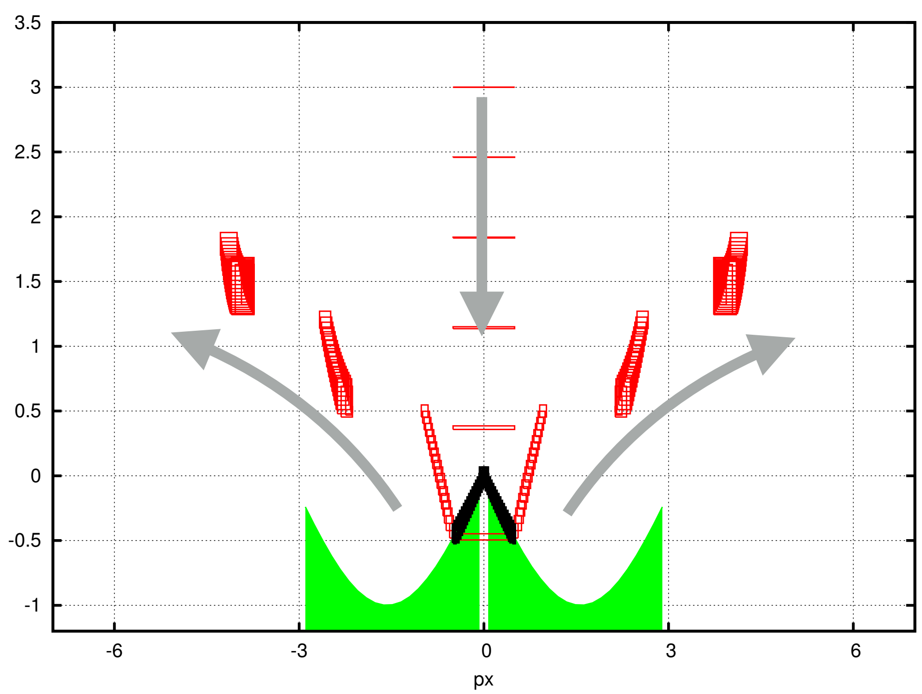
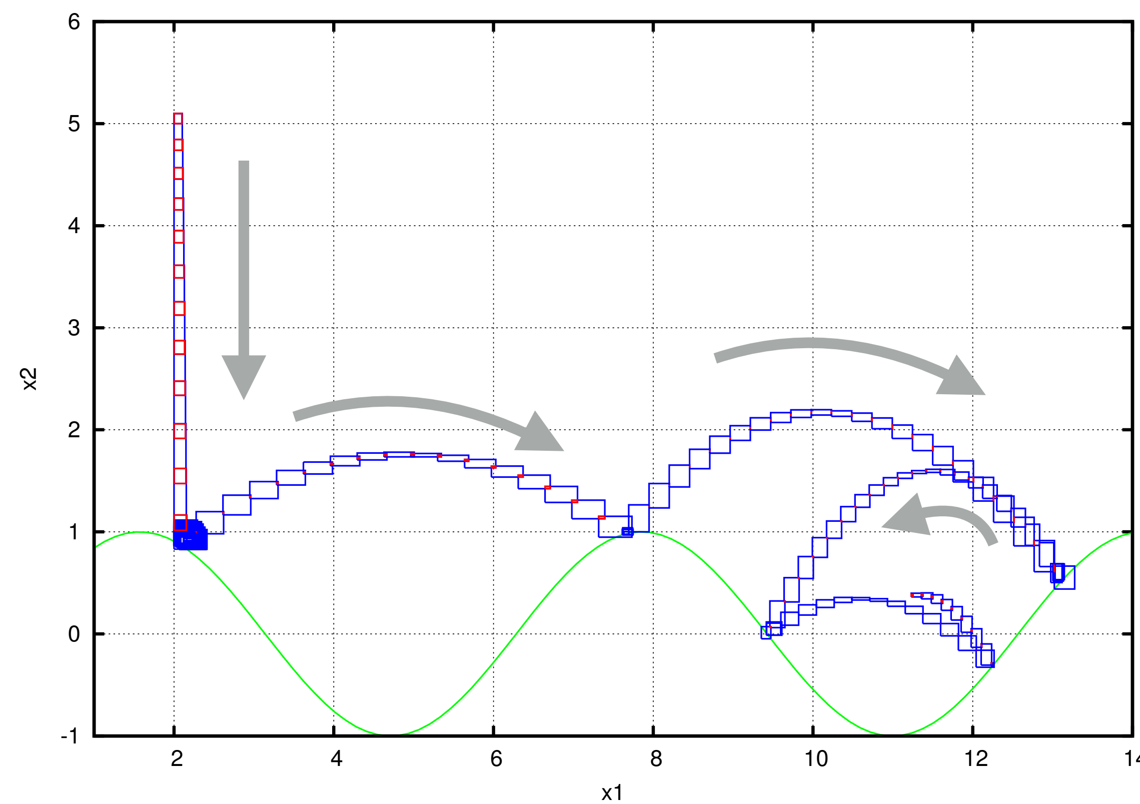


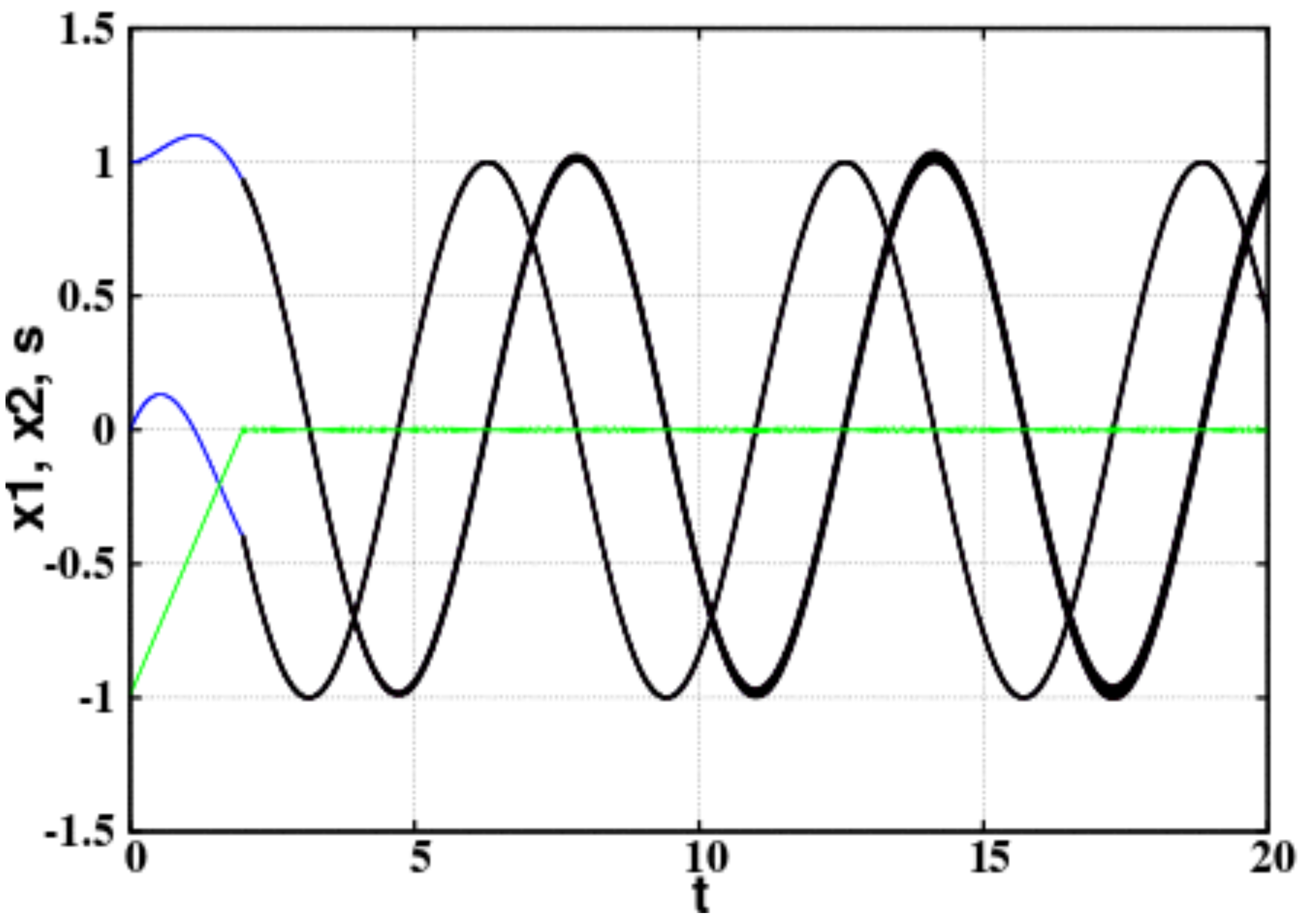




Hybrid Reachability Computation

Detecting and localizing events

- **Improved and enhanced version**
  - Impact of uncertainty on **sliding mode control**
    (Maïga, Ramdani, Travé-Massuyès, Combastel, IEEE TAC 2016)



Hybrid Reachability Computation

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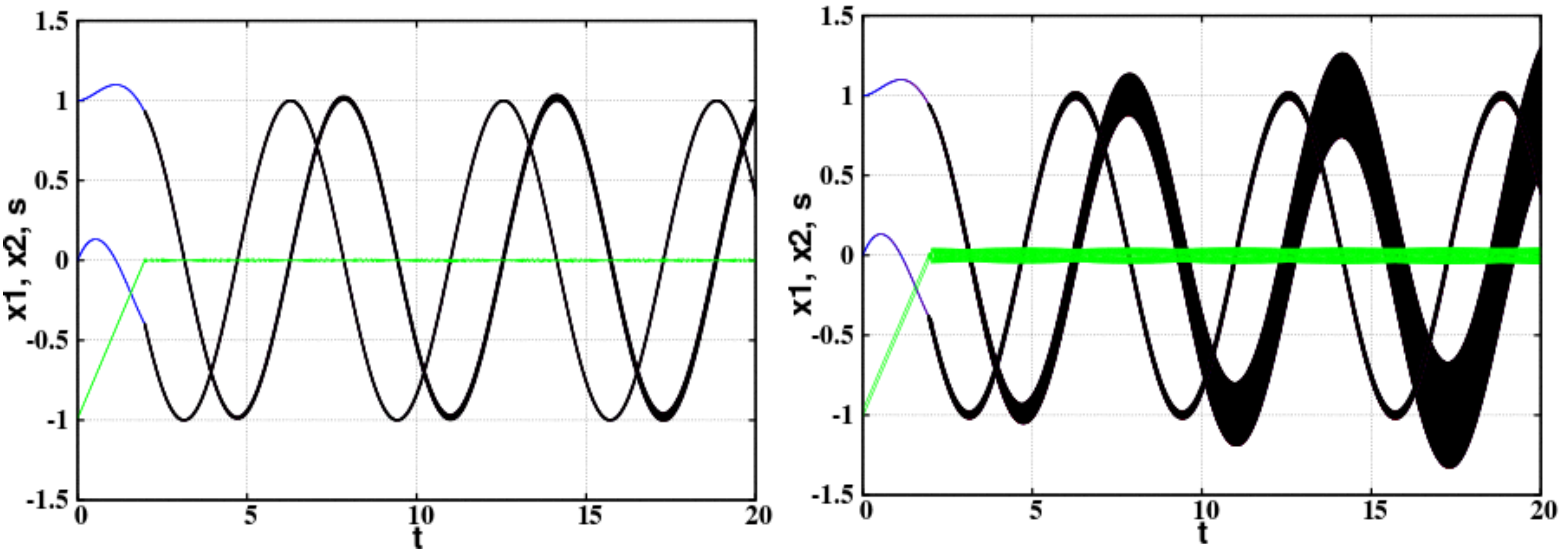
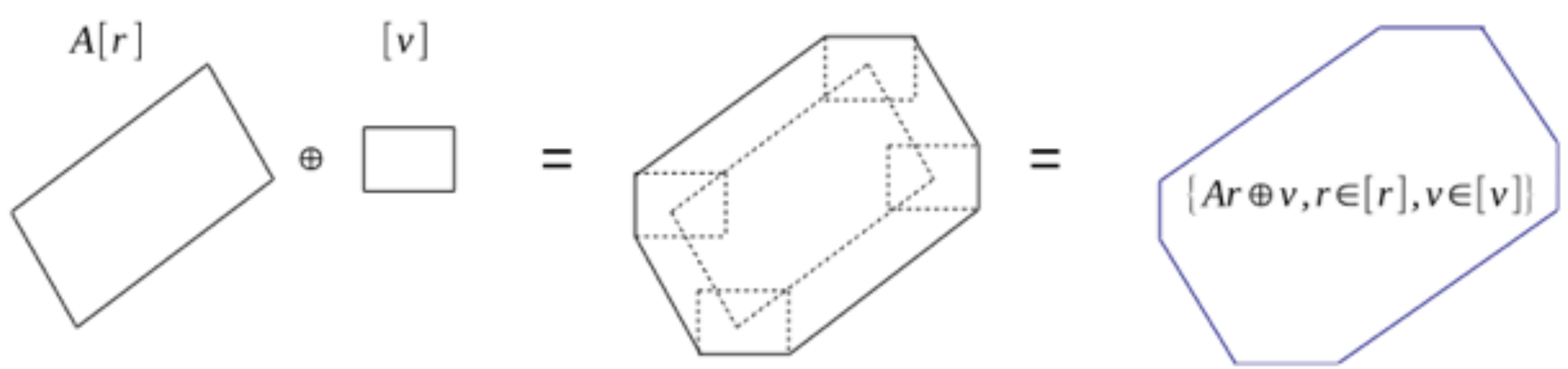


Solution trajectory tube

$$[\mathbf{x}](t) = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j], [\mathbf{p}]) + (t - t_j)^k \mathbf{f}^{[k]}([\tilde{x}_j], [\mathbf{p}])$$

Mean value form + Lohner's QR transformation method

$$\boxed{[\mathbf{x}](t) = A(t)[r](t) \oplus [v](t) \rightarrow \text{MSBP}}$$



$$\boxed{[\mathbf{x}](t) = c(t) \oplus R(t)\mathbf{B}^{2n}}$$ is a particular zonotope

$$\begin{aligned} c(t) &= A(t)mid([r](t)) + mid([v](t)), \\ R(t) &= (A(t)diagrad([r](t)) \mid diagrad([v](t))) . \end{aligned}$$

Zonotope of minimum size enclosing the intersection of a zonotope and a strip

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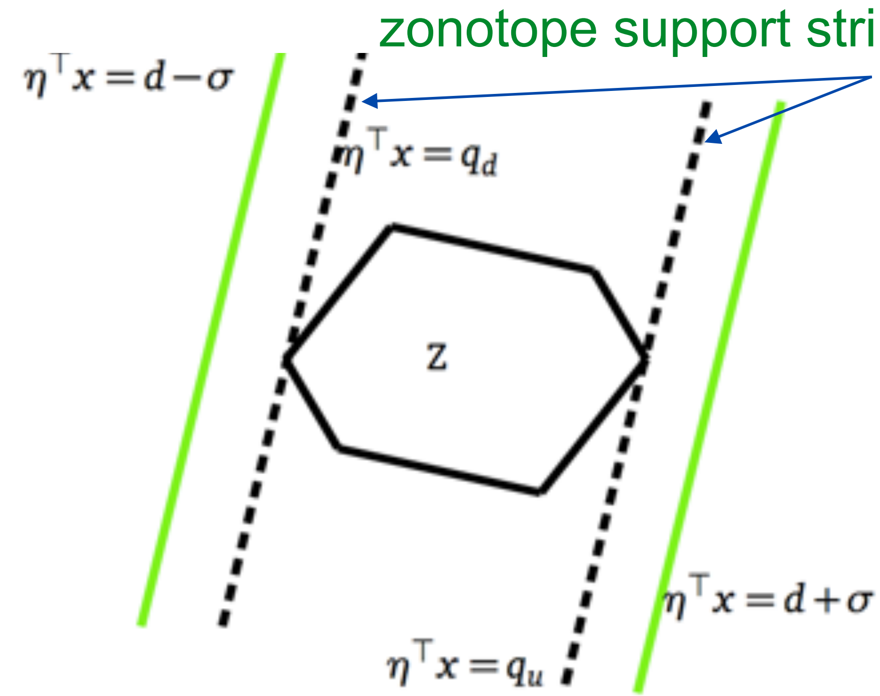
$\mathsf{Z}(c,R) = c \oplus R\boldsymbol{B}^p = \{c + Rx : x \in \boldsymbol{B}^p\}$ and $\mathcal{S} = \{x \in \mathbb{R}^n \mid |\eta^\top x - d| \leq \sigma\}$



Proposition 1 *(Vicino & Zappa, IEEE TAC 1996)*
Given zonotope $\mathsf{Z}(c,R)$ *and strip* $\mathcal{S}$, *the zonotope support strip is defined by*

$$\mathcal{S}_{\mathsf{Z}} = \{x \in \mathbb{R}^n \mid \rho_d \leq \eta^\top x \leq \rho_u\},$$
$$\rho_u = \eta^\top c + \|R^\top \eta\|_1$$
$$\rho_d = \eta^\top c - \|R^\top \eta\|_1$$

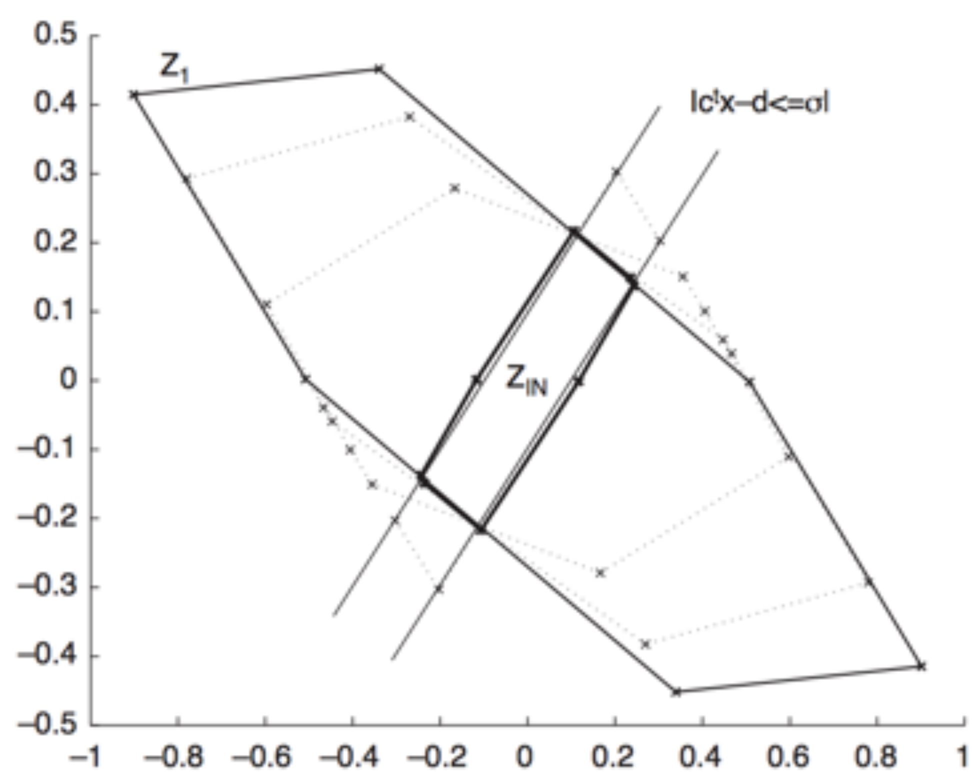
Theorem 1 *(Vicino & Zappa, IEEE TAC 1996)*

$$\mathsf{Z} = (c,R) \cap \mathcal{S} = \emptyset \quad \Leftrightarrow \quad (\rho_d > d + \sigma) \vee (\rho_u < d - \sigma)$$

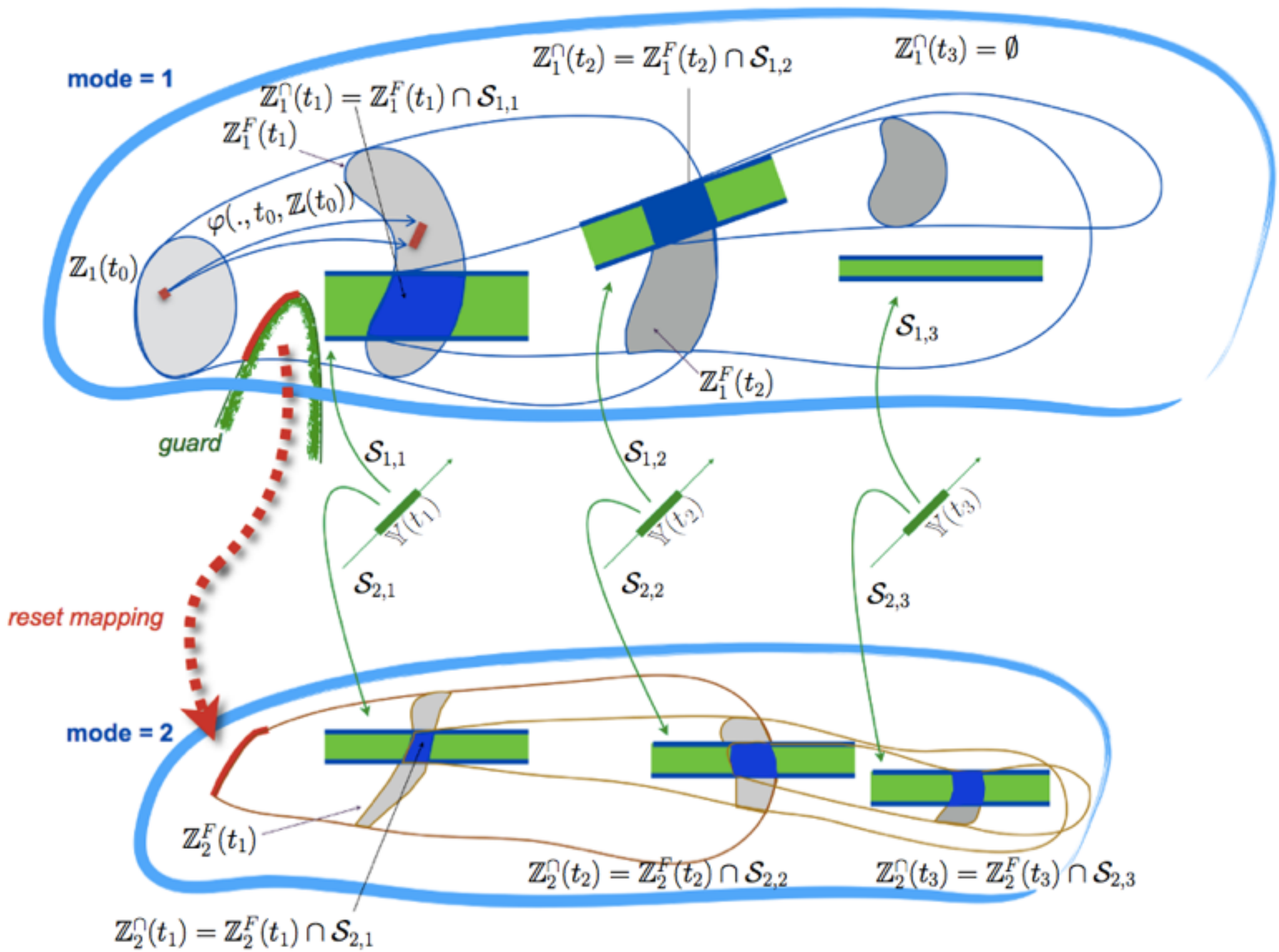
Proposition 2 *(Alamo et al., Automatica 2005)*
Given zonotope $\mathsf{Z}(c,R)$, *strip* $\mathcal{S}$, *and* $\lambda \in \mathbb{R}^n$, *define*

$$\hat{c}(\lambda) = c + \lambda(d - \eta^\top c),$$
$$\hat{R}(\lambda) = [(I - \lambda\eta^\top)R \quad | \quad \sigma\lambda],$$
$$\Rightarrow \quad \mathsf{Z} \cap \mathcal{S} = \hat{c}(\lambda) \oplus \hat{R}(\lambda)\mathbf{B}^{(p+1)}.$$

$\lambda^\star = \frac{RR^\top\eta}{\eta^\top RR^\top\eta + \sigma^2}$ minimizes the Frobenius norm of matrix $\hat{R}(\lambda)$.



Hybrid Reachability based Predictor Corrector approach



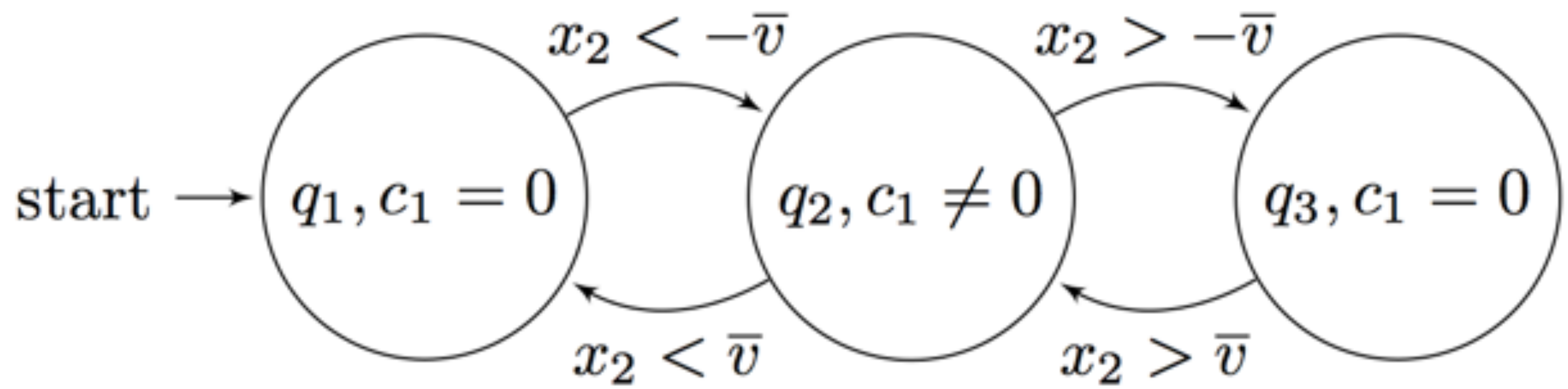
Outline

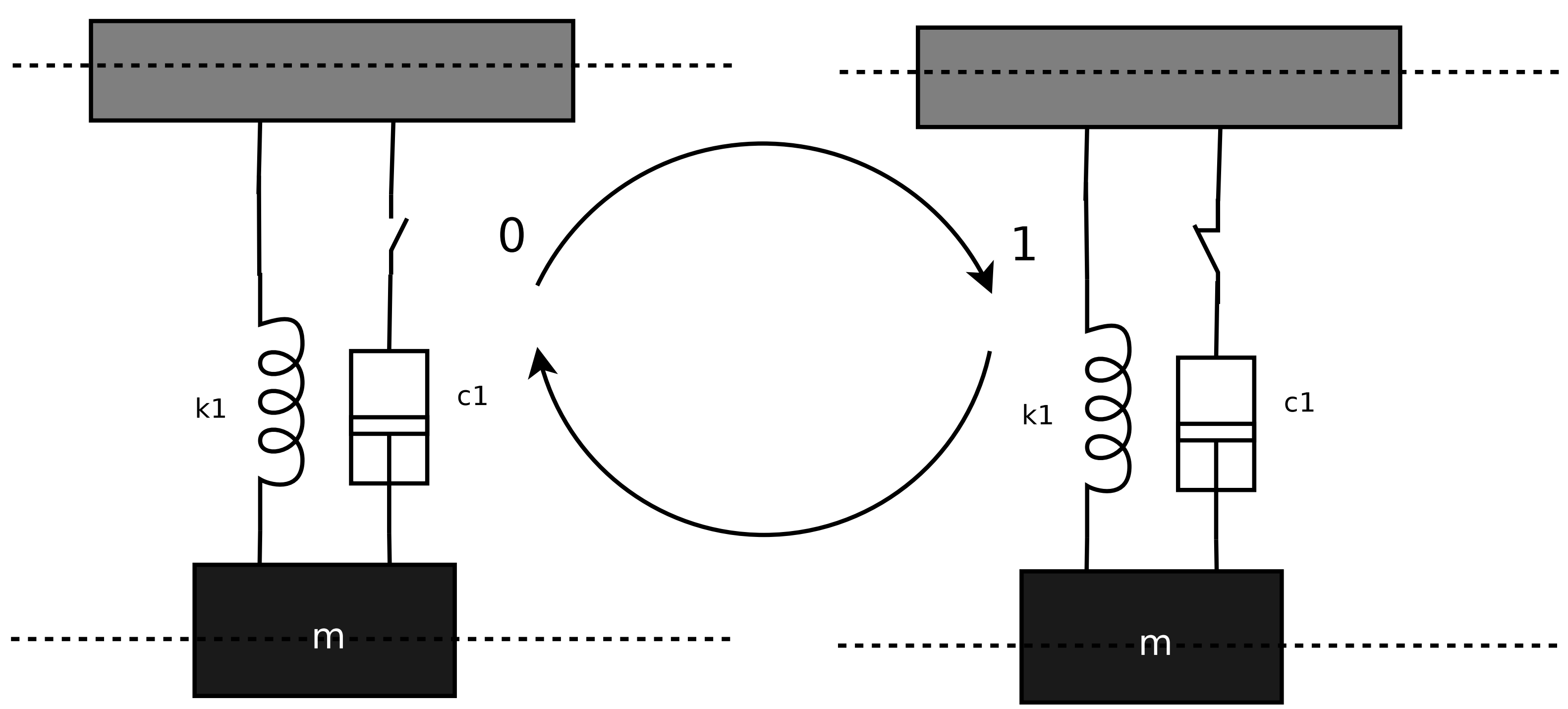
- Hybrid dynamical systems
- Set membership estimation
- Hybrid reachability based approach
- **Example**
- Research directions

Parameter identification

Hybrid Mass-Spring

- Velocity-dependent damping. Mode switching driven by velocity.

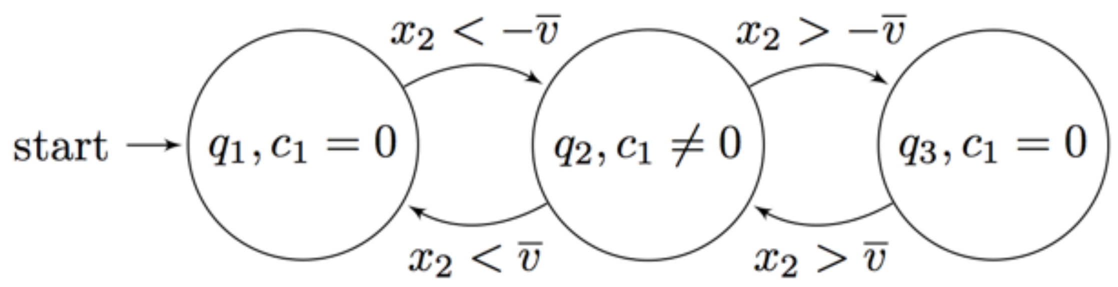


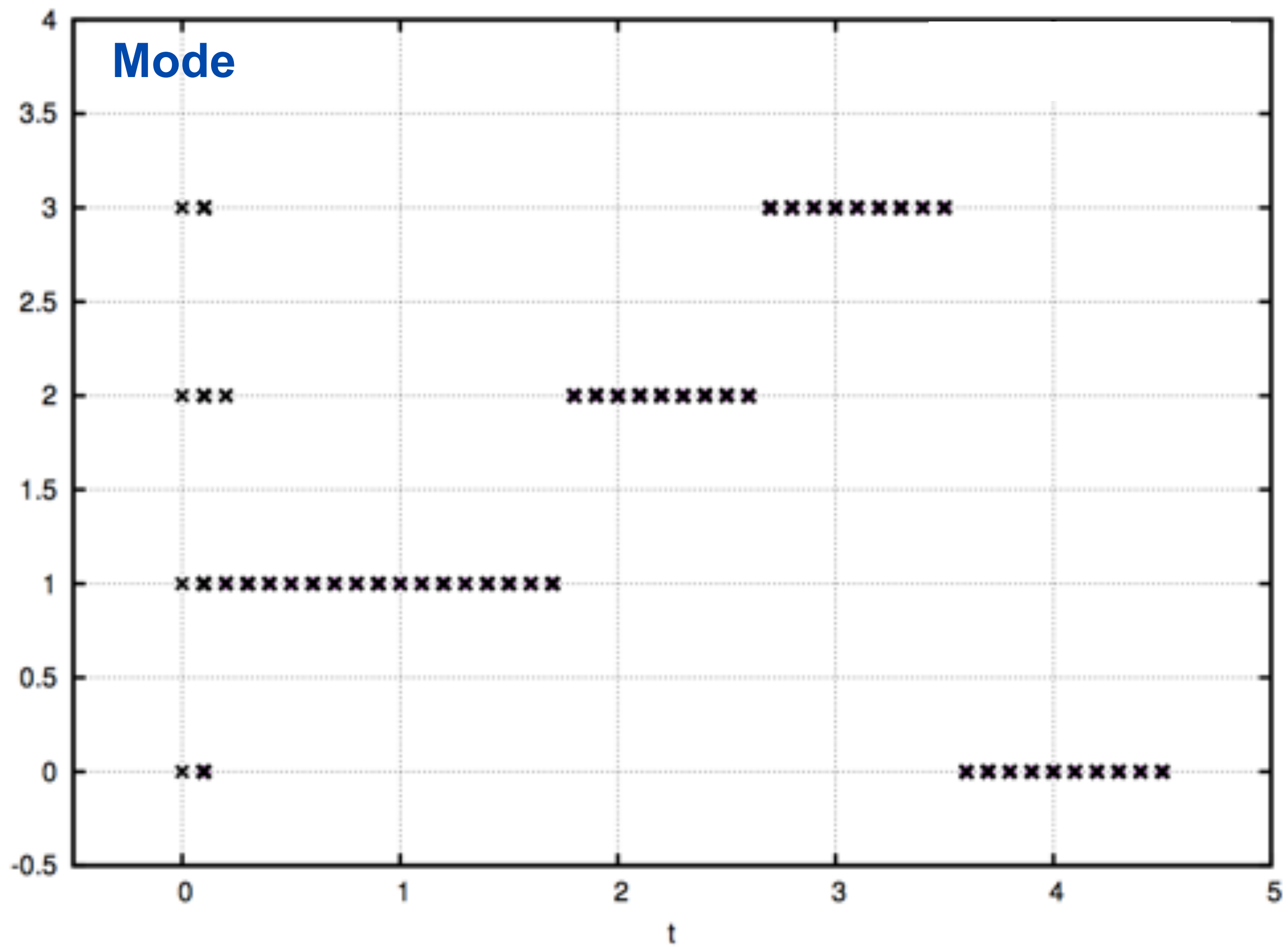


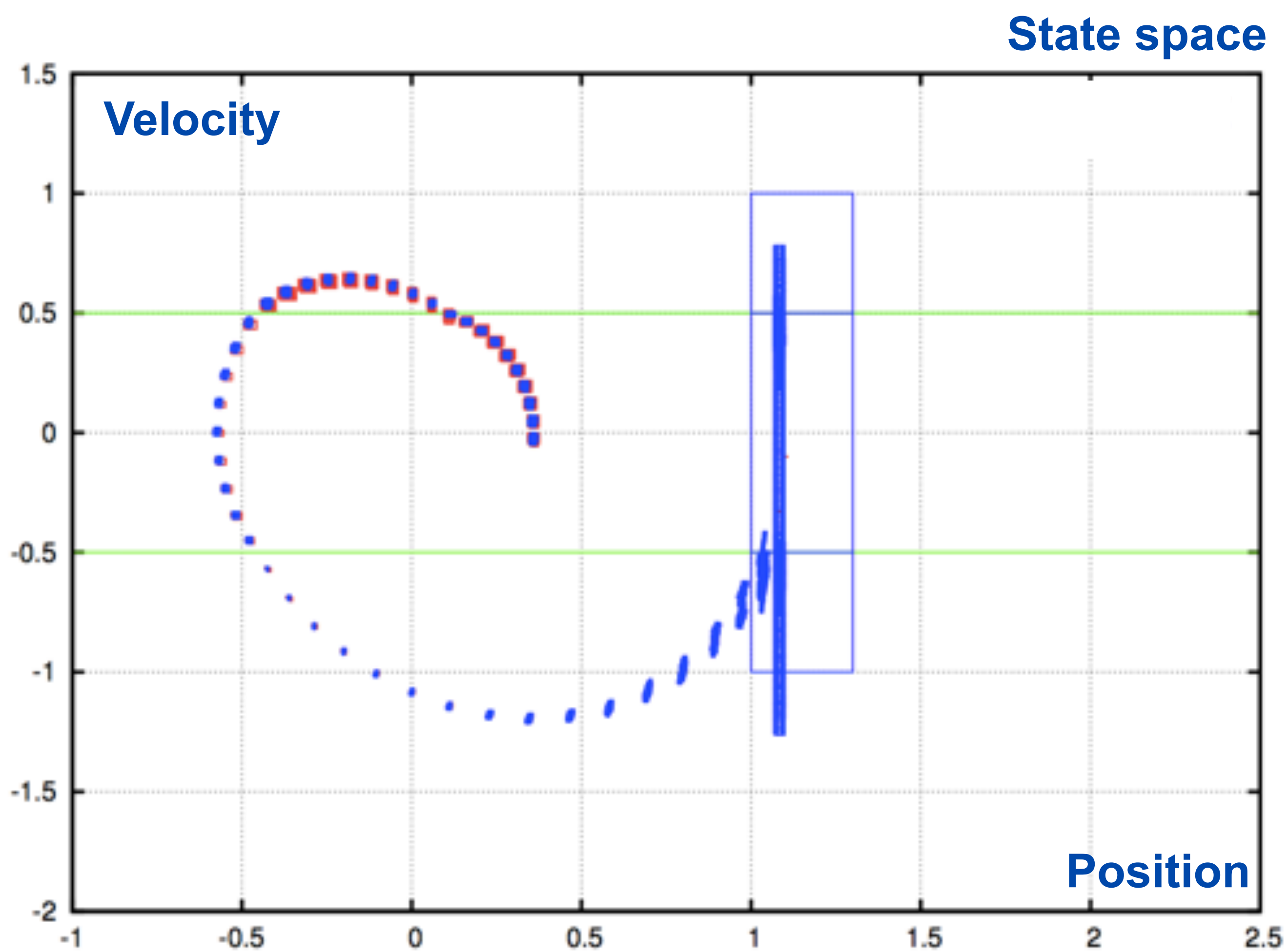
State Estimation

Hybrid Mass-Spring

- **Unknown initial mode.**
  - *CPU time approx.* *1m20s*







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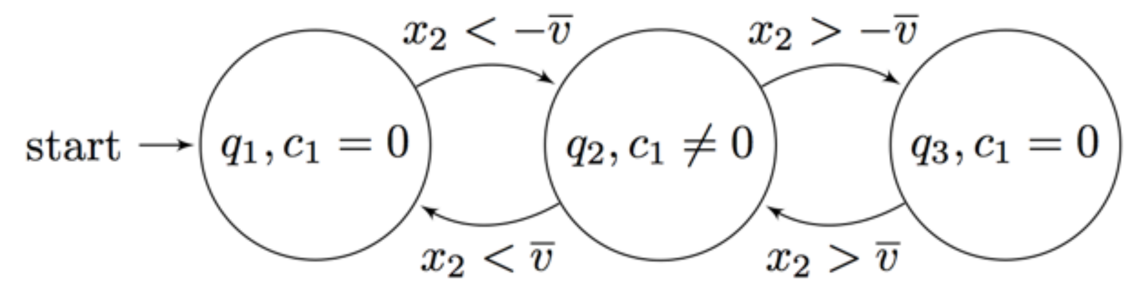

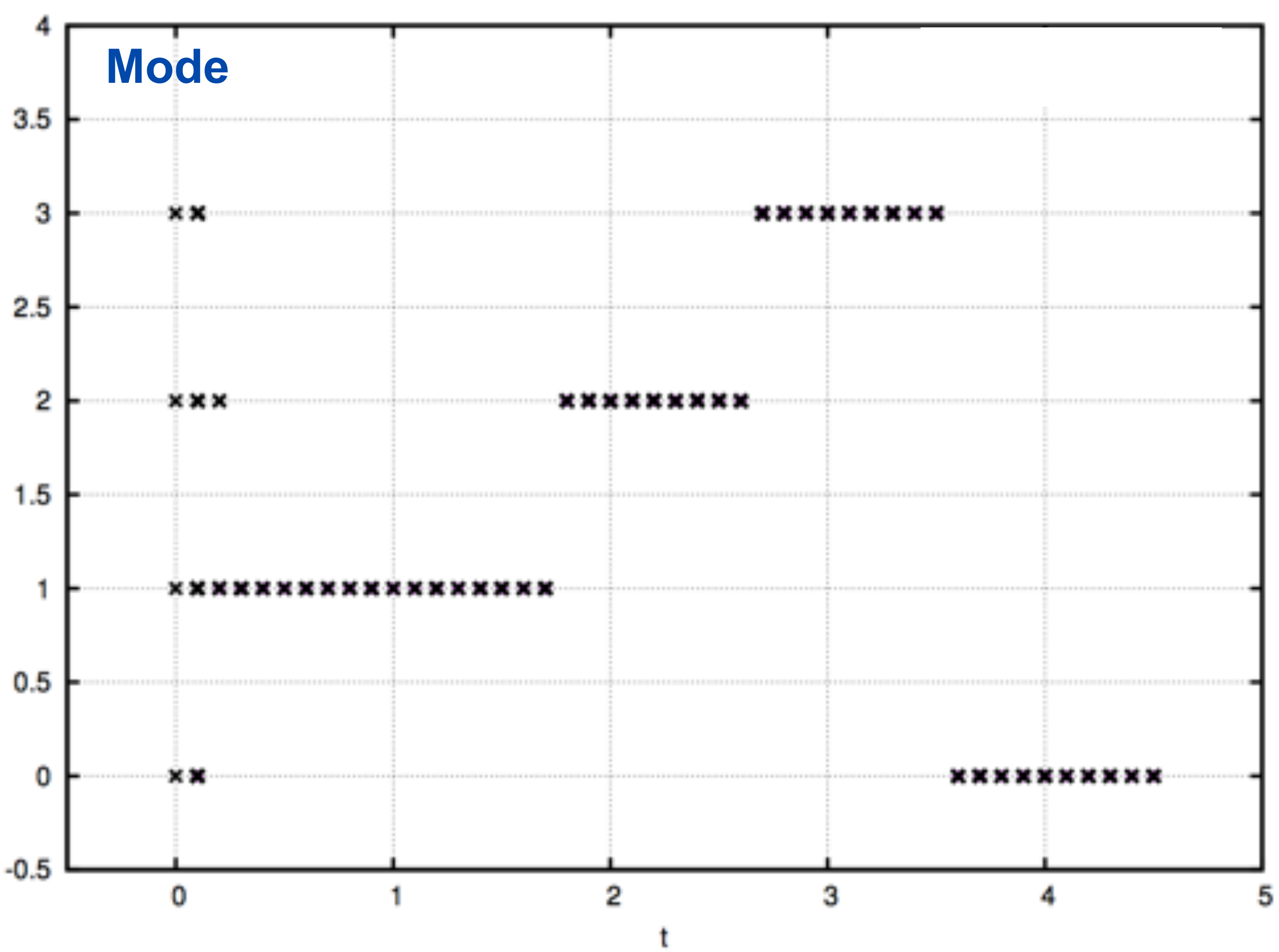

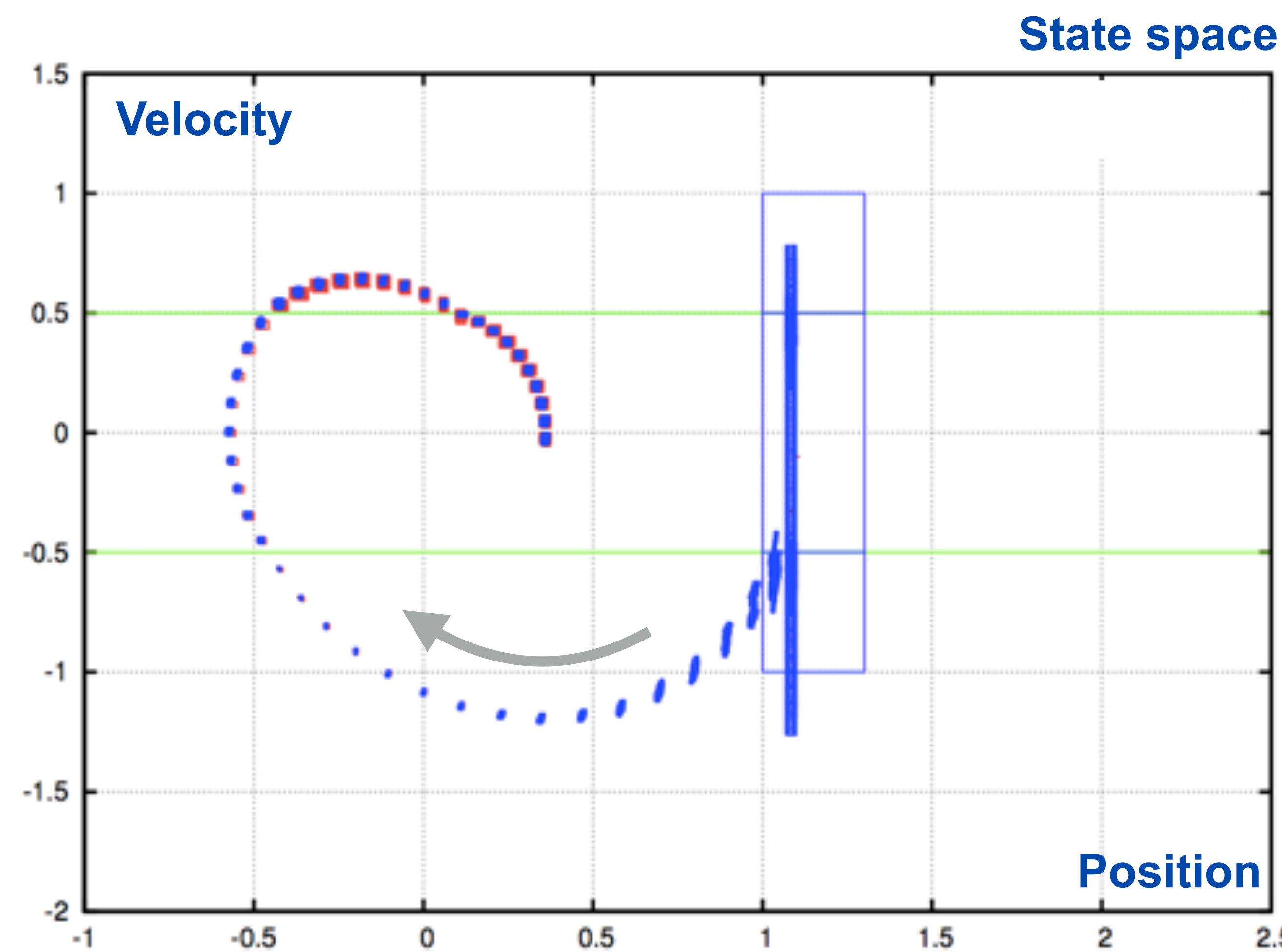

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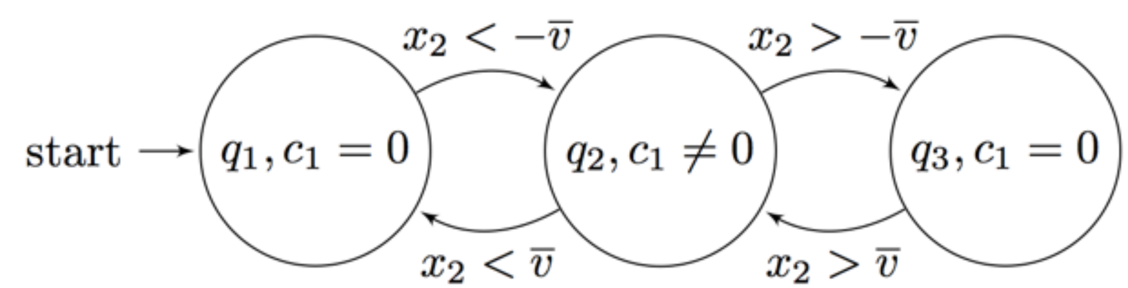

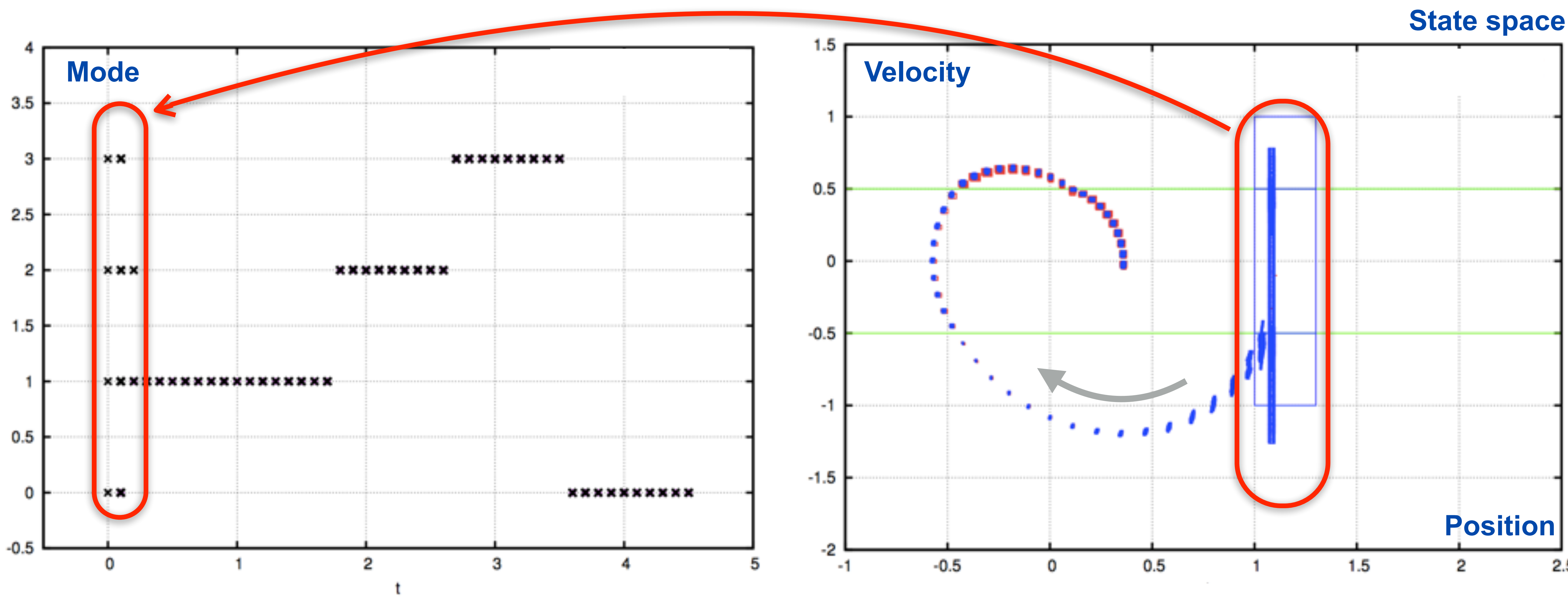

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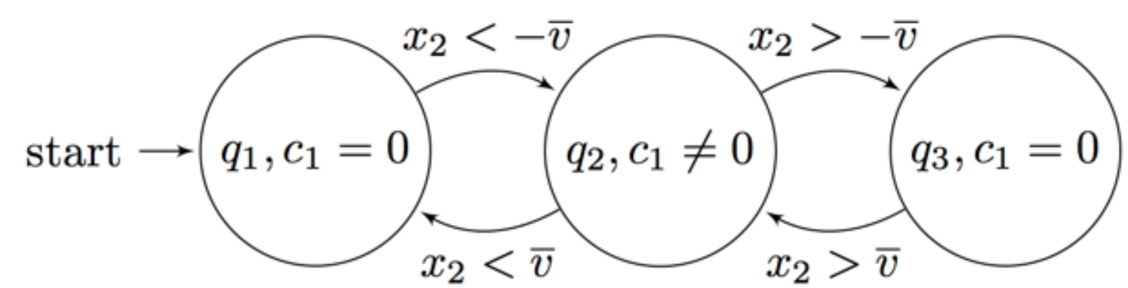

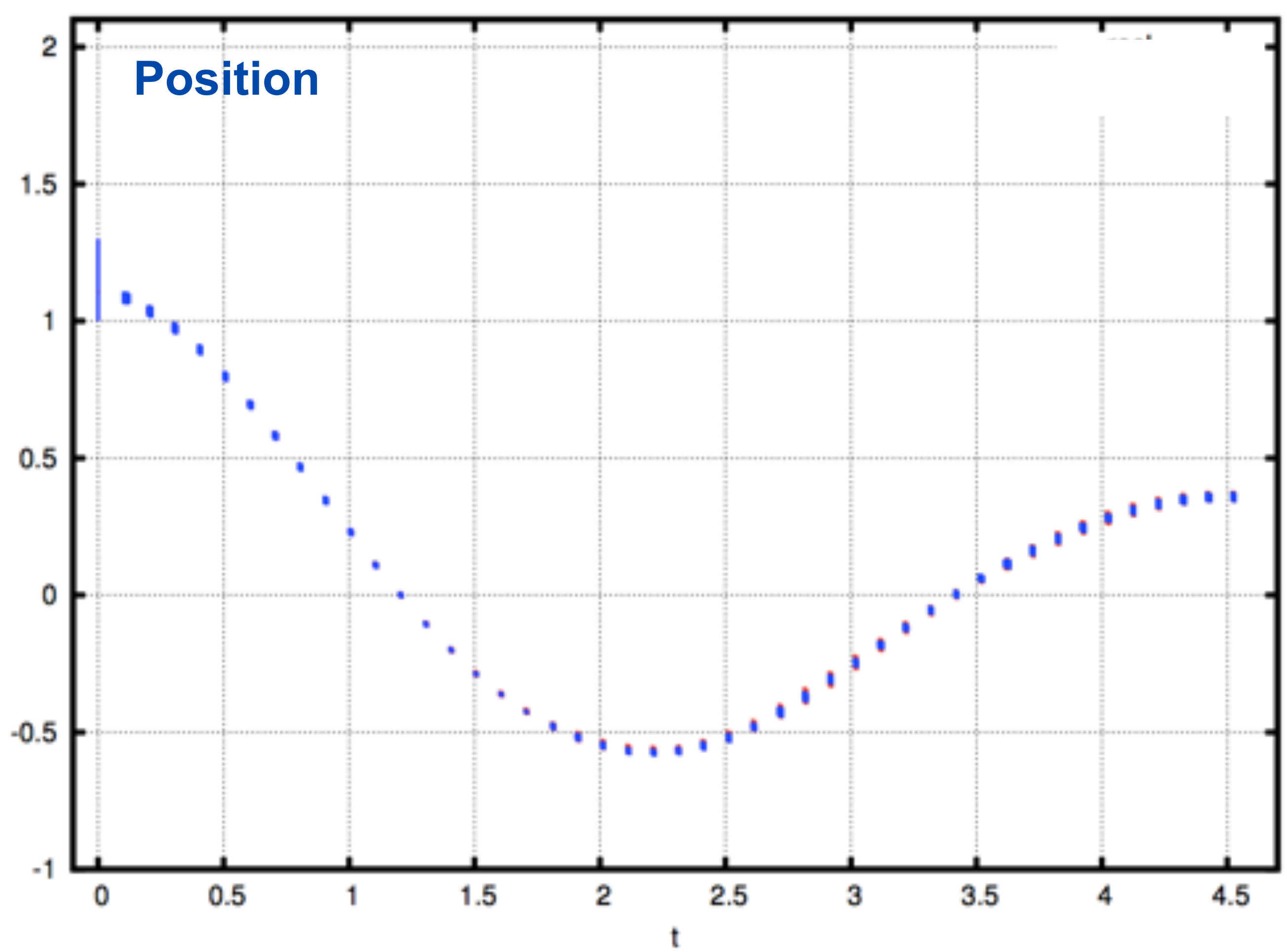

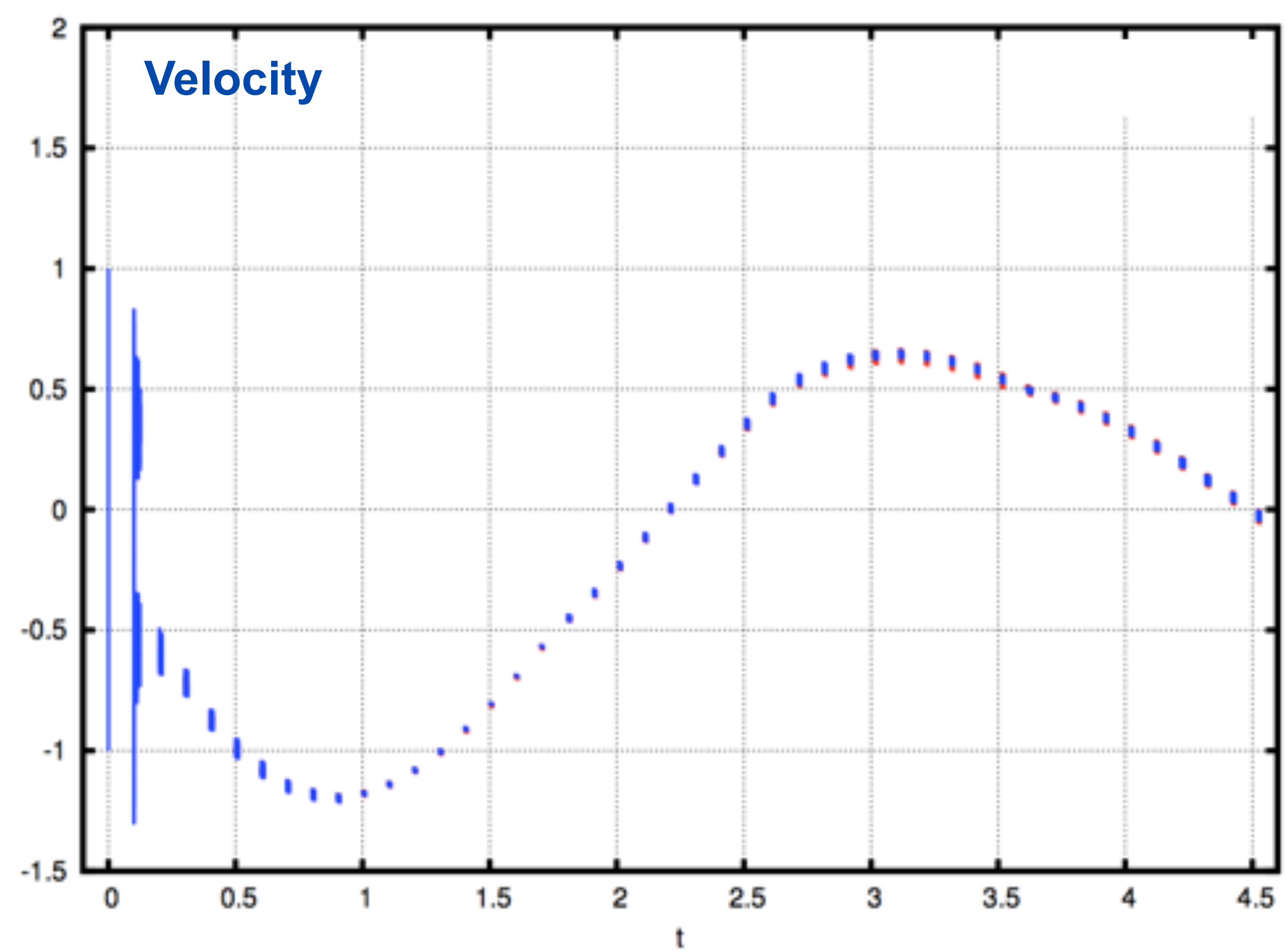

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Research directions

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  - To build upon a **hybrid reachability** approach
- Push forward set membership **estimation**
  - SM **hybrid** state estimation of nonlinear hybrid systems

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- Address SM estimation with **controlled** sampling
  - Event- & **Self-triggered** SM **hybrid** state estimation of nonlinear hybrid systems

Research directions

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  - SM **hybrid** state estimation of nonlinear hybrid systems
- Address SM estimation with **controlled** sampling
  - Event- & **Self-triggered** SM **hybrid** state estimation of nonlinear hybrid systems
- Combine with decision making
  - Application to actual hybrid systems, in robotics, smart buildings, personalized medicine

…

Focused References

- N. Ramdani and N. S.Nedialkov, Computing reachable sets for uncertain nonlinear hybrid systems using interval constraint propagation techniques, **Nonlinear Analysis: Hybrid Systems**, 5(2), pp.149-162, **2011**.
- A.Eggers, N.Ramdani, N.S.Nedialkov, M.Fränzle, Set-Membership Estimation of Hybrid Systems via SAT Mod ODE. in **IFAC SYSID 2012**. pp.440-445
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- M. Maïga, N. Ramdani, L. Travé-Massuyès, Robust fault detection in hybrid systems using set-membership parameter estimation, in **IFAC SafeProcess 2015**, Paris.
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