Set-based methods in programs and systems verification

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Automatic validation of numerical programs and systems

Validation of programs (discrete systems)

```
float xn,xnp1,residu,Input,Output,
      shouldbezero; int i=0;
float eps=0.00000001;
Input=FBETWEEN(16.0,16.002);
xn=1.0/Input; xnp1=xn;
residu=2.0*eps*(xn+xnp1)/(xn+xnp1);
while (fabs(residu) > eps) {
  xnp1=xn*(1.875+Input*xn*xn
         *(-1.25+0.375*Input*xn*xn));
  residu=2.0*(xnp1-xn)/(xn+xnp1);
  xn=xnp1;
  i++;
}
Output=1.0 / xnp1;
shouldbezero=Output-sqrt(Input);
```

- Does it crash? can we bound program variables? does it compute the square root? at what precision?
- For embedded systems, work on control code since the 2000s (FLUCTUAT, Astrée etc.) in particular using abstract interpretation (Cousot & Cousot 1977-) mostly on invariant properties
- Connections to set-based methods

Validation of hybrid systems (discrete and continuous systems)

- For embedded systems control, many properties only provable on closed-loop systems (with the physical environment)
- Even more connections to set-based methods (Taylor models as in e.g. Berz & Makino$\sim$1985)
- Connections with model-checking (Clarke, Emerson, Queille, Sifakis 1980) : generalisation of reachability and invariance properties (if some value is bounded for some time, can we reach some prescribed state?)

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Contents of the talk

Quick guided tour on our work on abstract interpretation of numerical programs

- Set-based methods and Abstract Interpretation, for validation of programs (here, just in real numbers!)
  - An introduction to abstract interpretation
  - Zonotopes for reachability, invariant synthesis, and functional proofs of algorithms

Some work in progress on abstract interpretation of hybrid systems

- Extension of zonotopic methods: inner- and outer- approximations of the set of solutions of uncertain ODEs
- Use for temporal verification, LTL, CTL and (abstract) model-checking

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The overall context, in the last 15 years or so

Computation of sets of reachable values of variables at any program points (FLUCTUAT)

- Need to bound real and finite precision values of variables, and the difference between them, decomposed on the provenance of these errors
- Accurate outer-approximation with affine forms
- Implemented in the FLUCTUAT analyzer for C programs

But how pessimistic are the results?

- Joint use of inner- and outer-approximations to characterize the quality of analysis results
  - Inner-approximation: sets of values of the outputs, that are sure to be reached for some inputs in the specified ranges.
  - Use of affine forms with generalized intervals as coefficients

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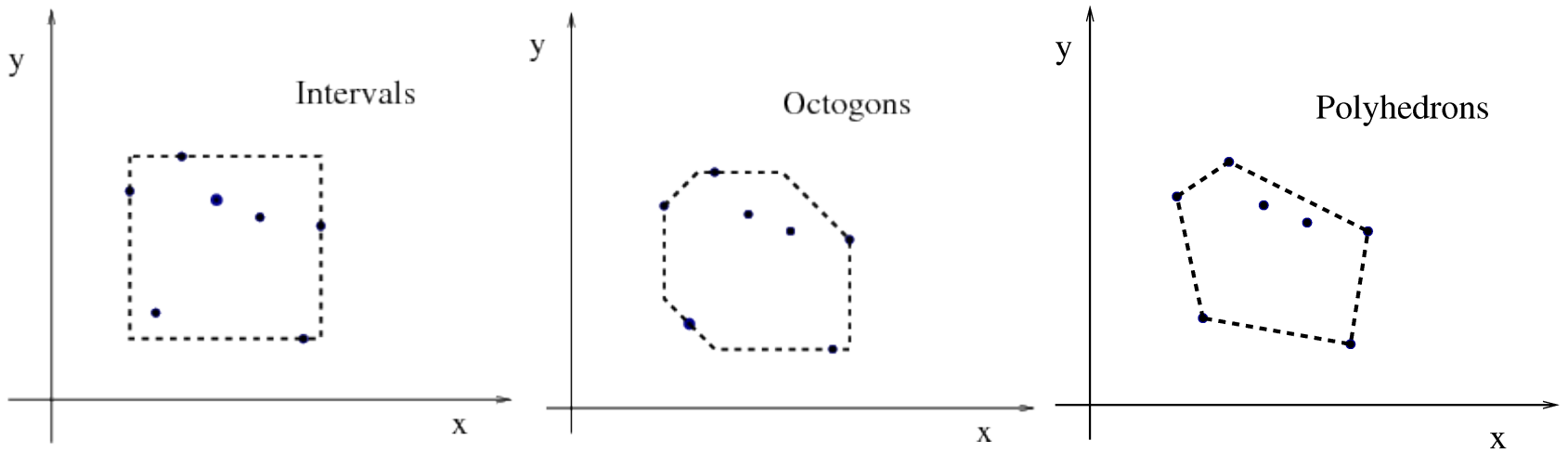
Static analysis by abstract interpretation (Cousot 77)

Basics

- Choose properties of interest (for instance values of variables)
- Outer-approximate them in an abstract lattice (partially ordered structure with least upper bounds/greatest lower bounds) whose elements are particular "tractable" sets
- Interpret computations in this lattice



Example in intervals

Abstract semantics of programs in intervals (for invariant generation)

- Program seen as a discrete dynamical system $X^{n+1} = F(X^n)$
  - based on a notion of control points in the program
  - equations describe how values of variables are collected at each control point, for all possible executions (collecting semantics)

```
void main() {
int x=[-100,50]; [1]
while [2] (x < 100) {
[3] x=x+1; [4]
} [5]
}
```

$$\begin{cases} x_0 = \top \\ x_1 = [-100, 50] \\ x_2 = x_1 \cup x_4 \\ x_3 =]-\infty, 99] \cap x_2 \\ x_4 = x_3 + [1, 1] \\ x_5 = [100, +\infty[\cap x_2 \end{cases}$$

Invariants generation = least fixed point computation

- The sets of possible values of variables at control points are invariants of F, computed as the least fixpoint of the system
- F monotonic on a complete lattice, least fixpoint exists

Invariants

Invariants and validation

- Invariants allow to conclude about the safety (for instance absence of run-time errors)
- E.g. we will find for :
  int x=[-100,50]; [1]
  while [2] (x $<$ 100) { [3] x=x+1; [4] } [5]
  $x_3 = [-100, 99]$ and the program will not run into an overflow

Computation of invariants as the least fixpoint $X = F(X)$

- Limit of the Kleene iteration (Jacobi/Gauss-Seidel like method)
  $X^0 = \bot$, $X^1 = F(X^0)$, ..., $X^{k+1} = X^k \cup F(X^k)$
  - with convergence acceleration to terminate in finite time
- An alternative: policy iteration (Newton-like method)

The least fixpoint is the best inductive invariant ($F(X) \subseteq X$) ...

- ... but invariants are not always inductive (in a given abstract domain)
- Search for a disjunction which is inductive: algorithm inspired from constraint programming (Mine and al. 2015 on boxes, B. Kabi's talk)

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Affine Arithmetic (Comba & Stolfi 93) for real-numbers abstraction

Affine forms

- Affine form for variable x:

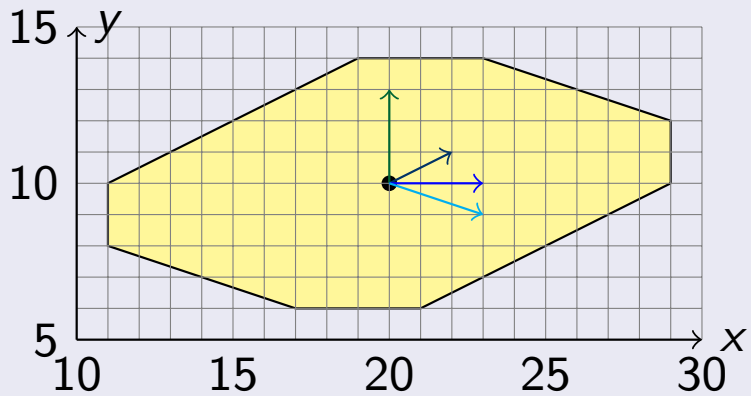
$$\hat{x} = x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n,\ x_i \in \mathbb{R}$$

  where the ε_i are symbolic variables (*noise symbols*), with value in $[-1, 1]$.
- Sharing ε_i between variables expresses implicit dependency
- Interval concretization of affine form $\hat{x}$:

$$\left[x_0 - \sum_{i=0}^{n} |x_i|, x_0 + \sum_{i=0}^{n} |x_i|\right] = x_0 + [-\|(x_i)\|_1, \|(x_i)\|_1]$$

Geometric concretization as zonotopes (center symmetric polytopes)

$$\begin{aligned} \hat{x} &= 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4 \\ \hat{y} &= 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4 \end{aligned}$$



Huge litterature - (dual) generator representation of a polytope!

Basic arithmetic operations

- Assignment $x := [a, b]$ introduces a noise symbol:

$$\hat{x} = \frac{(a+b)}{2} + \frac{(b-a)}{2}\varepsilon_i.$$

- Addition/subtraction are exact:

$$\hat{x} + \hat{y} = (x_0 + y_0) + (x_1 + y_1)\varepsilon_1 + \ldots + (x_n + y_n)\varepsilon_n$$

- Non linear operations : approximate linear form, new noise term bounding the approximation error

$$\hat{x} \times \hat{y} = x_0 y_0 + \sum_{i=0}^{n} (x_0 y_i + x_i y_0)\varepsilon_i + \left(\sum_{1 \leq i \neq j \leq n} | x_i y_j | \right) \varepsilon_{n+1}$$

  (better formulas including SDP computations of the new term)
- Close to Taylor models of order 1: low time complexity! and easy to implement on a finite-precision machine

Reminder!

Need to define an order relation and interpret set-theoretic operations such as $\cup$ and $\cap$ (or at least outer-approximations), as for e.g. :

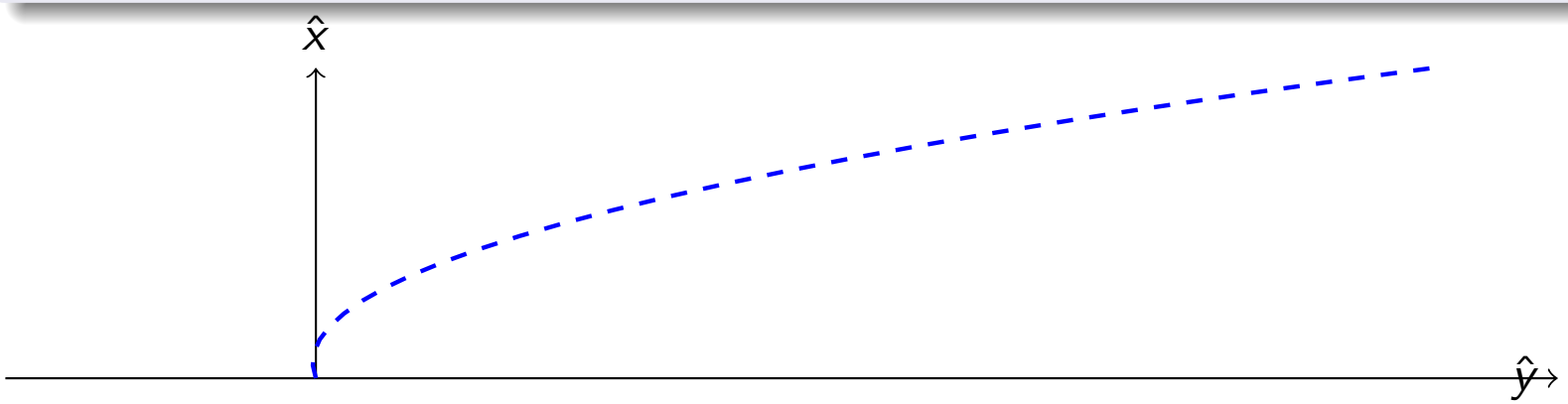
$$\begin{cases} x_1 &= [-100, 50] \\ x_2 &= x_1 \cup x_4 \\ x_3 &=]-\infty, 99] \cap x_2 \\ x_4 &= x_3 + [1, 1] \\ x_5 &= [100, +\infty[\cap x_2 \end{cases}$$

Note

- We are actually abstracting input-output relationships, not just the image of functions (see Arxiv 2008 & 2009 & FMSD 2016)
- This is the major difference with respect to classical work on zonotopes (Comba & Stolfi, Girard etc.)

A simple example: functional interpretation

```
real x = [0,10];
real y = x*x - x;
```



Abstraction of x: $x = 5 + 5\varepsilon_1$
Abstraction of function $x \to y = x^2 - x$ as

$$y = 32.5 + 50\varepsilon_1 + 12.5\eta_1$$

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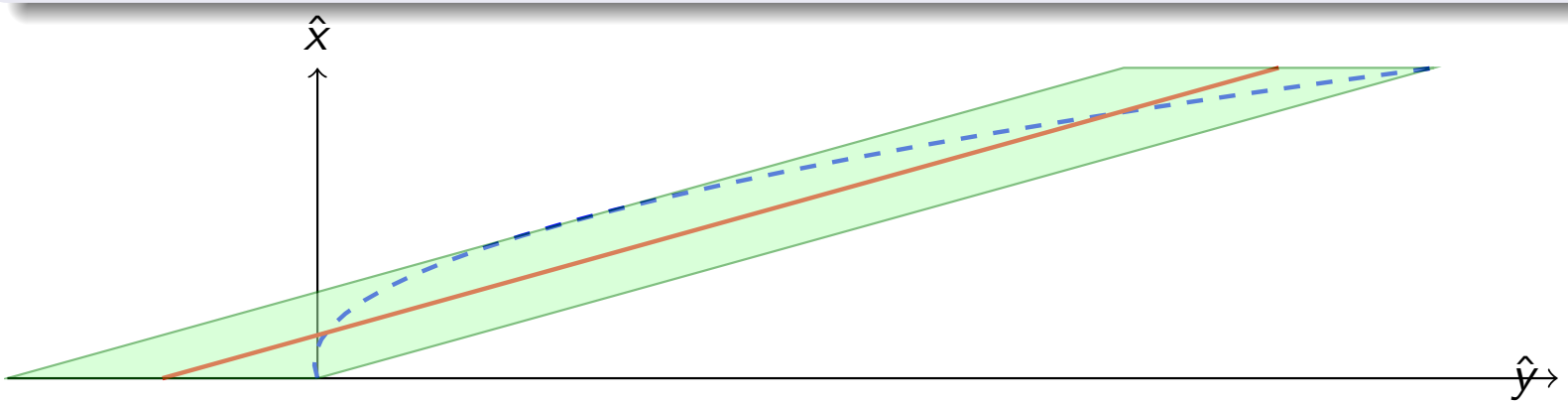
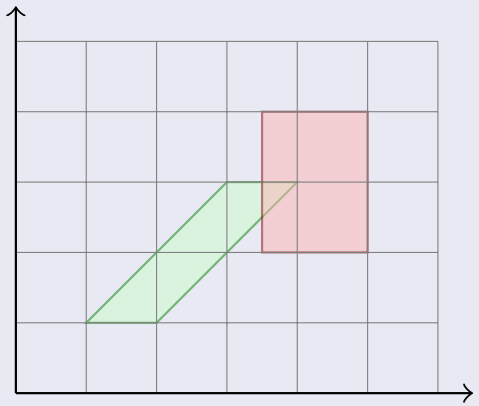


Abstraction of x: $x = 5 + 5\varepsilon_1$
Abstraction of function $x \to y = x^2 - x$ as

$$\begin{aligned} y &= 32.5 + 50\varepsilon_1 + 12.5\eta_1 \\ &= -17.5 + 10x + 12.5\eta_1 \end{aligned}$$

Set operations on affine sets / zonotopes: meet

Test interpretation: intersection of zonotopes with guards are not zonotopes!



Interpreting tests

- Translate the condition on noise symbols
- Abstract domain for the noise symbols: intervals, octagons, etc.
- Test interpretation is (interval) constraint propagation

Example

```
real x = [0,10]; real y = 2*x;
if (y >= 10) y = x;
```

- Affine forms before tests: $x = 5 + 5\varepsilon_1$, $y = 10 + 10\varepsilon_1$
- In the `if` branch: constraint $\varepsilon_1 \geq 0$

Set operations on affine sets / zonotopes: meet

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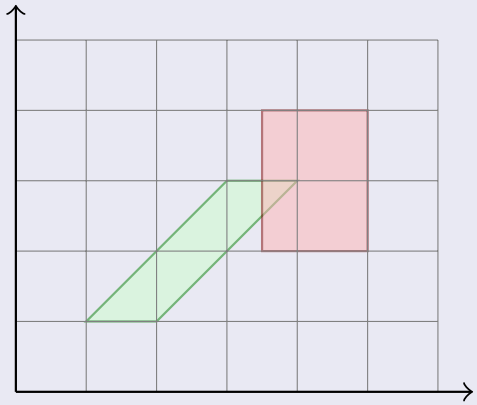

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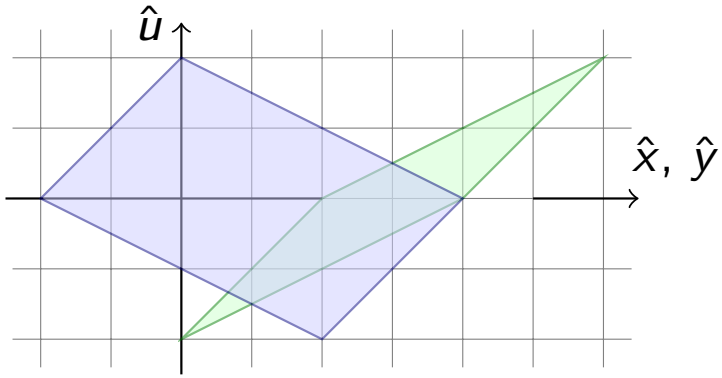
When going to finite precision analysis

- Constraints give condition on idealized (in reals) and machine (in floating-point numbers) paths to be taken by an execution
- Unstable test detection is a constraint satisfaction problem

Join operator

$$\left(\begin{array}{l}\hat{x} = 3 + \varepsilon_1 + 2\varepsilon_2 \\ \hat{u} = 0 + \varepsilon_1 + \varepsilon_2\end{array}\right) \cup \left(\begin{array}{l}\hat{y} = 1 - 2\varepsilon_1 + \varepsilon_2 \\ \hat{u} = 0 + \varepsilon_1 + \varepsilon_2\end{array}\right) = \left(\begin{array}{lcl}\hat{x} \cup \hat{y} & = & 2 + \varepsilon_2 + 3\eta_1 \\ \hat{u} & = & 0 + \varepsilon_1 + \varepsilon_2\end{array}\right)$$



Construction (low complexity!: $\mathcal{O}(n \times p)$)

- Keep "minimal common dependencies"

$$z_i = \underset{x_i \wedge y_i \leq r \leq x_i \vee y_i}{\operatorname{argmin}} |r|, \ \forall i \geq 1$$

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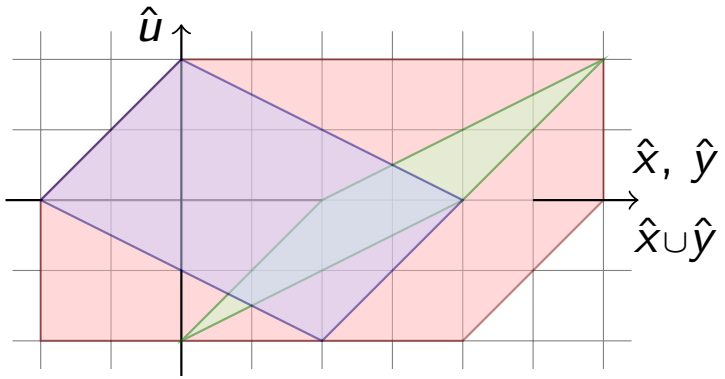

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$$z_i = \underset{x_i \wedge y_i \leq r \leq x_i \vee y_i}{\operatorname{argmin}} |r|, \ \forall i \geq 1$$

- For each dimension, concretization is the interval union of the concretizations: $\gamma(\hat{x} \cup \hat{y}) = \gamma(\hat{x}) \cup \gamma(\hat{y})$
- A more precise upper bound: NSAD 2012

Convergence of fixpoint computation: from concrete to abstract

General result on recursive linear filters, pervasive in embedded programs:

$$x_{k+n+1} = \sum_{i=1}^{n} a_i x_{k+i} + \sum_{j=1}^{n+1} b_j e_{k+j}, \quad e_l \in [m, M]$$

- Concrete scheme has bounded outputs iff zeros of $x^n - \sum_{i=0}^{n-1} a_{i+1} x^i$ have modulus stricty lower than 1.
- Then our Kleene iteration (with some initial unfolding p and uncyclic unfolding q) converges towards a finite outer-approximation of the outputs

$$\hat{X}_i = \hat{X}_{i-1} \cup F^q(E_i, \ldots, E_{i-k}, \hat{X}_{i-1}, \ldots, \hat{X}_{i-k})$$

in finite time

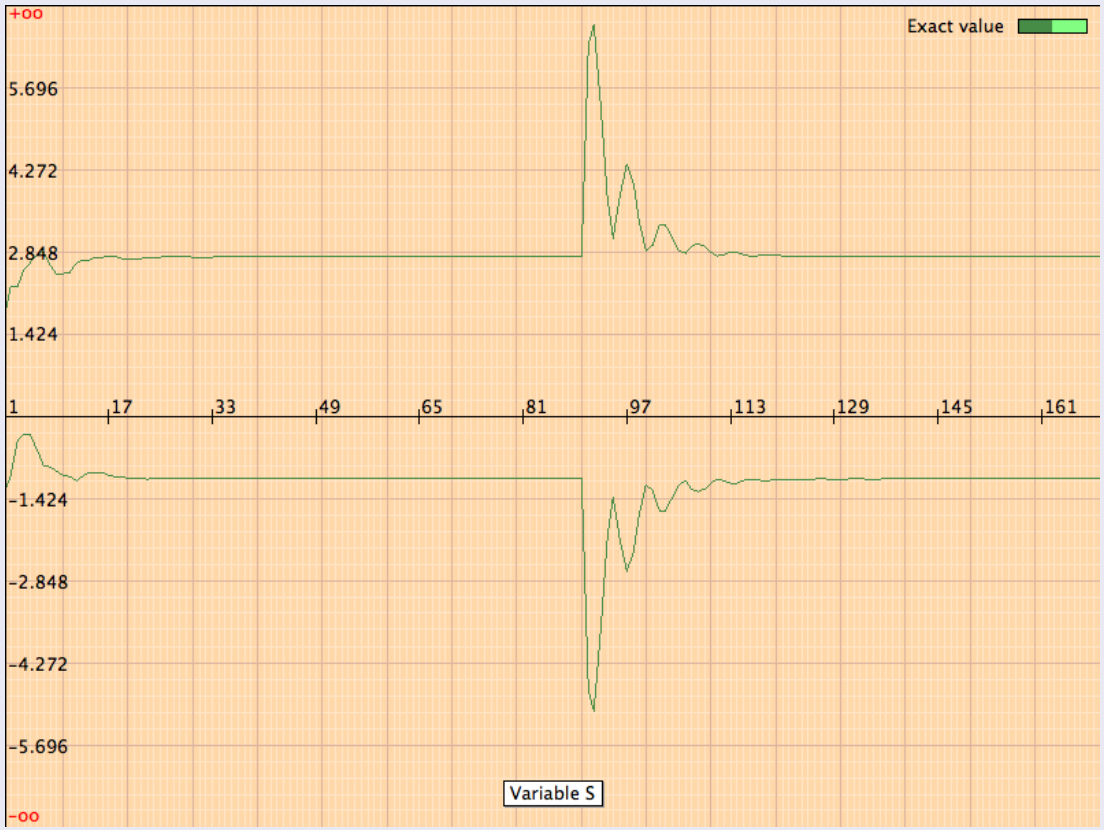
  - The abstract scheme is a perturbation (by the join operation) of the concrete scheme
  - Proof uses: for each dimension $\gamma(\hat{x} \cup \hat{y}) = \gamma(\hat{x}) \cup \gamma(\hat{y})$ and F^q is contracting "enough" for some q

Generalization to some recurrent polynomial schemes

Illustration: a simple order 2 filter

$S_{n+2} = 0.7E_{n+2} - 1.3E_{n+1} + 1.1E_n + 1.4S_{n+1} - 0.7S_n$

- after initial unfolding (10)+first cyclic unfolding (80) - first join
- after first join, perturbation of the original numerical scheme
- then second cyclic unfolding, contracting back: second join and post-fixpoint



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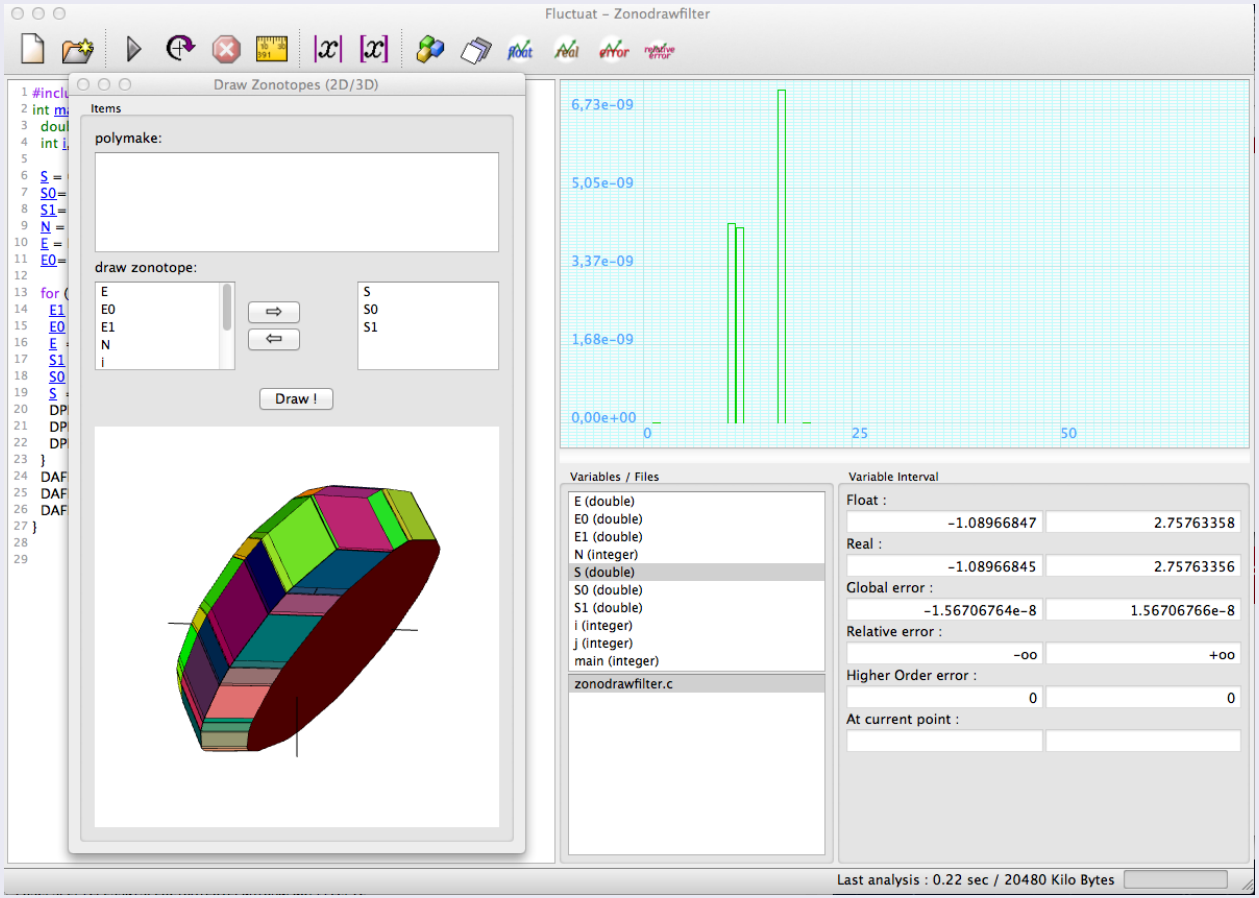
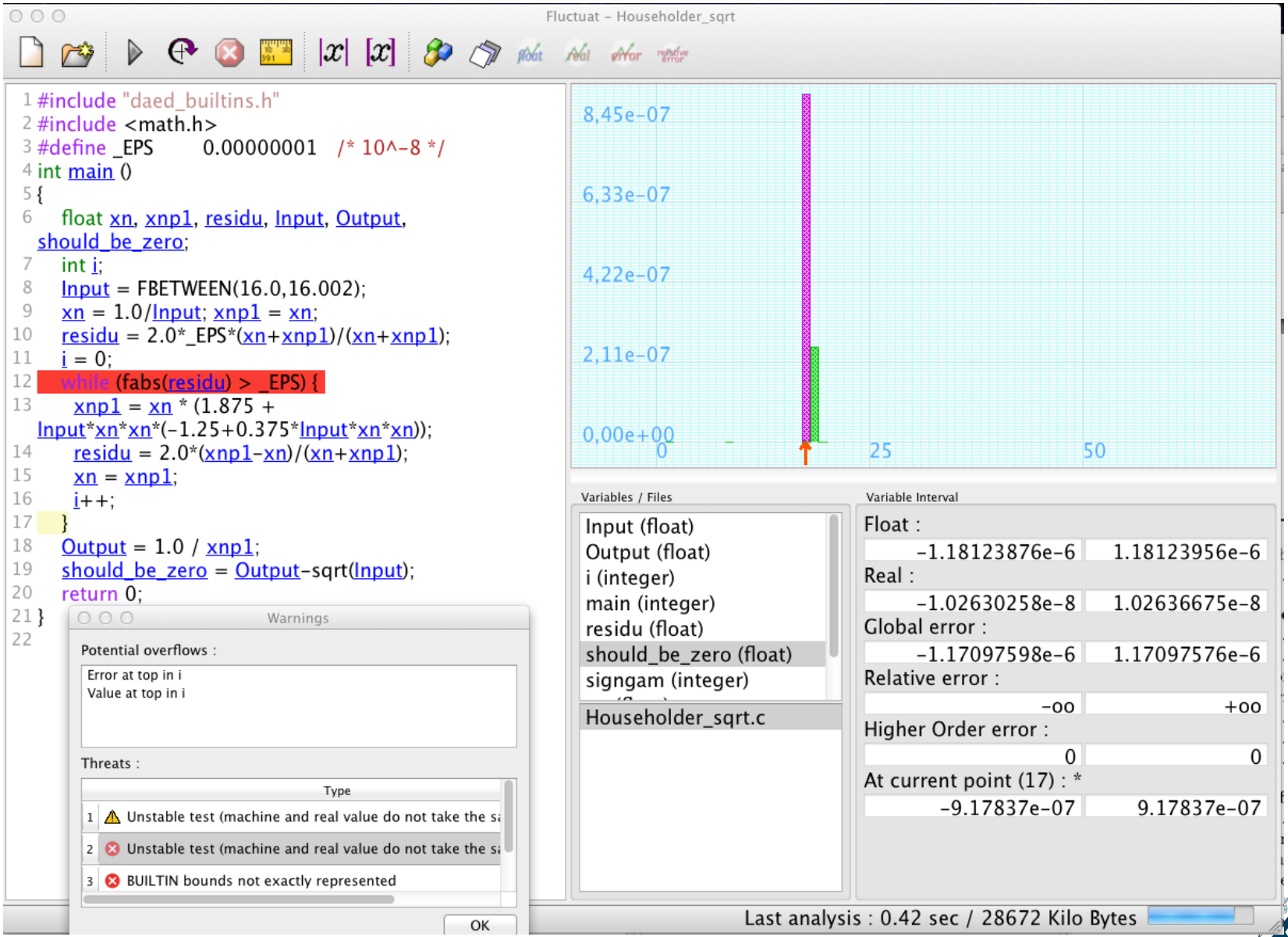
Fixpoint (polyhedral outer-approximation of the ellipsoidal invariant) below:



Fluctuat : also properties of finite precision arithmetic (here Householder)



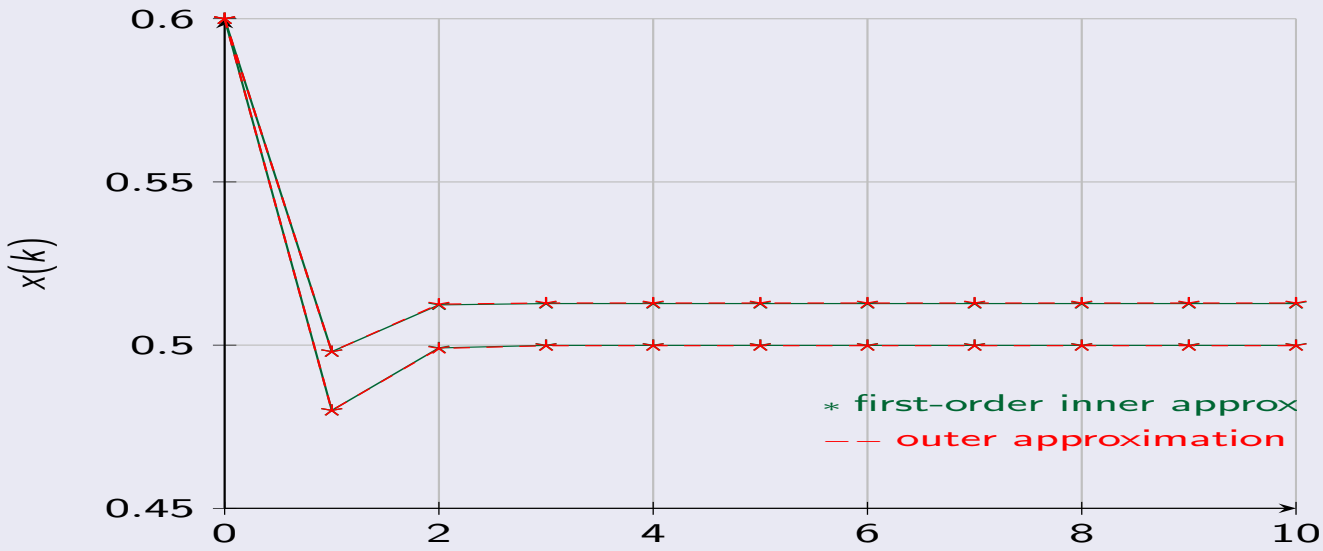
Inner-approximations?

Starting point

- Assert the quality of abstractions by looking at inner and outer approx
- Falsify properties
- Provide lower/upper bounds for convergence of numerical schemes

Newton algorithm for $a \to 1/a$, stop when $|x_{n+1} - x_n| < 5.10^{-4}$

- Outer approximation: stopping criterion always satisfied after 4 iter ($|x(4) - x(3)| \subseteq [-2.6\,10^{-4}, 2.6\,10^{-4}]$).
- Inner approximation: some inputs falsify criterion on first 3 iterations ($[-7.7\,10^{-4}, -4.1\,10^{-4}] \subseteq x(3) - x(2)$).



How to do this?

Inner approximation much more difficult than outer approximation (some pointers)

- Modal arithmetics (Kaucher 1973, Markov 1992, Goldsztejn 2005), Goldsztejn and Jaulin 2006
- Linear case for ODEs [Kurzhanski-Varaiya HSCC 2000, Althoff et al. CDC 2007, Kanade et al. CAV 2009]
- Simulation-based local inner approximations [Nghiem et al. HSCC 2010]
- Box bisections [Goldsztejn-Jaulin Reliable Computing 2010, Mullier-Goubault-Kieffer-Putot RC 2013]
- Parallelepipeds [Goldsztejn-Hayes SCAN 2006]
- Order 0 generalized affine forms [Goubault-Putot SAS 2007], order 1 [Goubault-Kieffer-Mullier-Putot HSCC 2014]

Will be useful in the rest of the talk...

- Check general temporal properties (TCTL in particular) on hybrid systems
- By a combination of inner- and outer- approximations

Generalized intervals

General bounds $\mathbf{K} = \{[a, b],\ a \in \mathbb{R}, b \in \mathbb{R}\}$ ("improper" if $a > b$)

Kaucher arithmetic (only when no dependencies between arguments!)

All proper intervals : outer-approx $(\forall x \in [x])\,(\exists z \in [z])\,(f(x) = z)$
All improper intervals : inner-approx $(\forall z \in \text{pro } [z])\,(\exists x \in \text{pro } [x])(f(x) = z)$.

Remedy : Mean-value theorem (à la Goldsztejn 2005)+affine arithmetic

Let $f : \mathbb{R}^n \to \mathbb{R}$ differentiable, $(t_1, \ldots, t_n)$ a point in $[-1, 1]^n$ and $\mathbf{\Delta_i}$ such that

$$\left\{ \frac{\partial f}{\partial \varepsilon_i}(\varepsilon_1, \ldots, \varepsilon_i, t_{i+1}, \ldots, t_n),\ \varepsilon_i \in [-1, 1] \right\} \subseteq \mathbf{\Delta_i}.$$

$$\text{Then } \tilde{f}(\varepsilon_1, \ldots, \varepsilon_n) = f(t_1, \ldots, t_n) + \sum_{i=1}^{n} \mathbf{\Delta_i}(\varepsilon_i - t_i),\ \text{means}$$

- if $\tilde{f}(\boldsymbol{\varepsilon_1^*}, \ldots, \boldsymbol{\varepsilon_n^*})$, $\boldsymbol{\varepsilon_i^*} = [1, -1]$, computed with Kaucher arithmetic, is an improper interval, then pro $\tilde{f}(\boldsymbol{\varepsilon_1^*}, \ldots, \boldsymbol{\varepsilon_n^*})$ is an inner-approx of $f(\boldsymbol{\varepsilon_1}, \ldots, \boldsymbol{\varepsilon_n})$.
- $\tilde{f}(\boldsymbol{\varepsilon_1}, \ldots, \boldsymbol{\varepsilon_n})$, $\boldsymbol{\varepsilon_i} = [-1, 1]$, is an outer-approx of $f(\boldsymbol{\varepsilon_1}, \ldots, \boldsymbol{\varepsilon_n})$.

Generalized affine forms

- The generalized mean-value theorem defines generalized affine forms: for $f : \mathbb{R}^n \to \mathbb{R}$,

$$f^{\varepsilon}(t_1, \ldots, t_n) + \sum_{i=1}^{n} \mathbf{\Delta_i}(\varepsilon_i - t_i),$$

  where $\left\{ \frac{\partial f^{\varepsilon}}{\partial \varepsilon_i}(\varepsilon),\ \varepsilon \in [-1,1]^n \right\} \sqsubseteq \mathbf{\Delta_i}$.
- We want an inductive computation of these forms on arithmetic expressions

Order 0 forms

- The partial derivatives $\mathbf{\Delta_i}$ are evaluated with intervals
- Example: $f(x) = x^2 - x$, $x \in [2,3]$, thus
  $f^{\varepsilon}(\epsilon_1) = (2.5 + 0.5\varepsilon_1)^2 - (2.5 + 0.5\varepsilon_1)$.
  We get $\tilde{f}^{\varepsilon}(\varepsilon_1) = 3.75 + [1.5, 2.5]\varepsilon_1$, that can be interpreted as:

$$pro(3.75 + [1.5, 2.5][1, -1]) \subseteq f([-1,1]) \subseteq 3.75 + [1.5, 2.5][-1,1]$$

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$$pro([5.25, 4.25]) \subseteq f([-1,1]) \subseteq [1.25, 6.25]$$

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$$[4.25, 5.25] \subseteq f([-1,1]) \subseteq [1.25, 6.25]$$

- Solves the single-occurence limitation but not quite the dependency problem

Generalized affine forms

- The generalized mean-value theorem defines generalized affine forms: for $f : \mathbb{R}^n \to \mathbb{R}$,

$$f^{\varepsilon}(t_1, \ldots, t_n) + \sum_{i=1}^{n} \mathbf{\Delta_i}(\varepsilon_i - t_i),$$

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- We want an inductive computation of these forms on arithmetic expressions

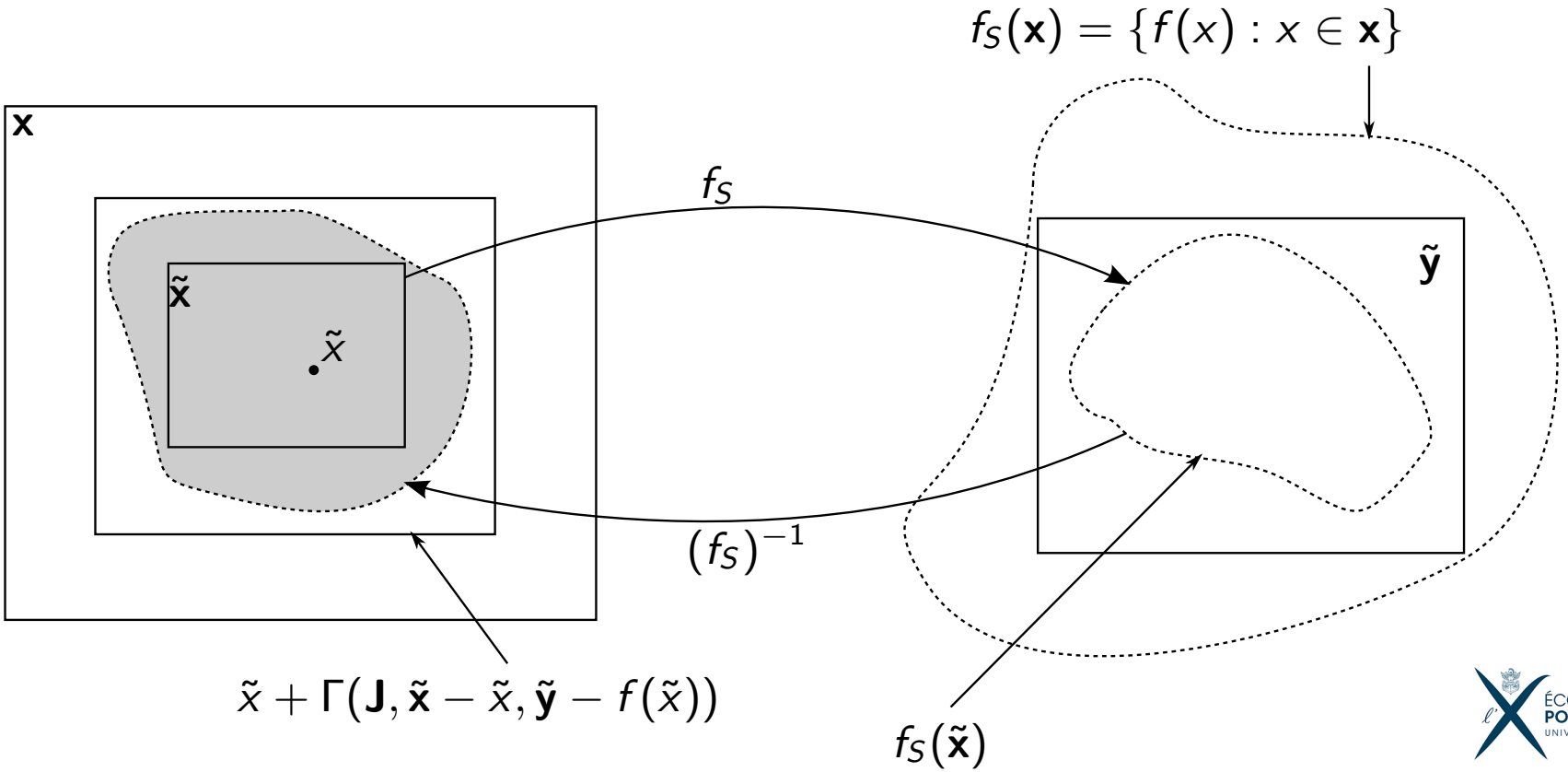
Order 1 forms

- Inductive computations with zonotopic outer-approximations of quantities and partial derivatives $\mathbf{\Delta_i}$: more precise that order 0
- When computing the inner range of a scalar function as above, we use only the interval range $\mathbf{\Delta_i}$
- But in general we have $f : \mathbb{R}^n \to \mathbb{R}^p$ and thus vectors of generalized affine forms
- Order 1 forms code some dependency between the components of f or f^{ε} : also allows us to define joint inner range

Joint inner range of a vector function

Algorithm to compute a set of boxes proved to be in the image of f:

- Based on input set bisection + a sufficient condition for a box $\tilde{\mathbf{y}}$ to be in range$(f, \mathbf{x})$.
- Only needs an outer approximation of the Jacobian of f
- Goldzstejn-Jaulin 2010 ($f : \mathbb{R}^n \to \mathbb{R}^n$), MGKP 2013 (extension $f : \mathbb{R}^n \to \mathbb{R}^p$)



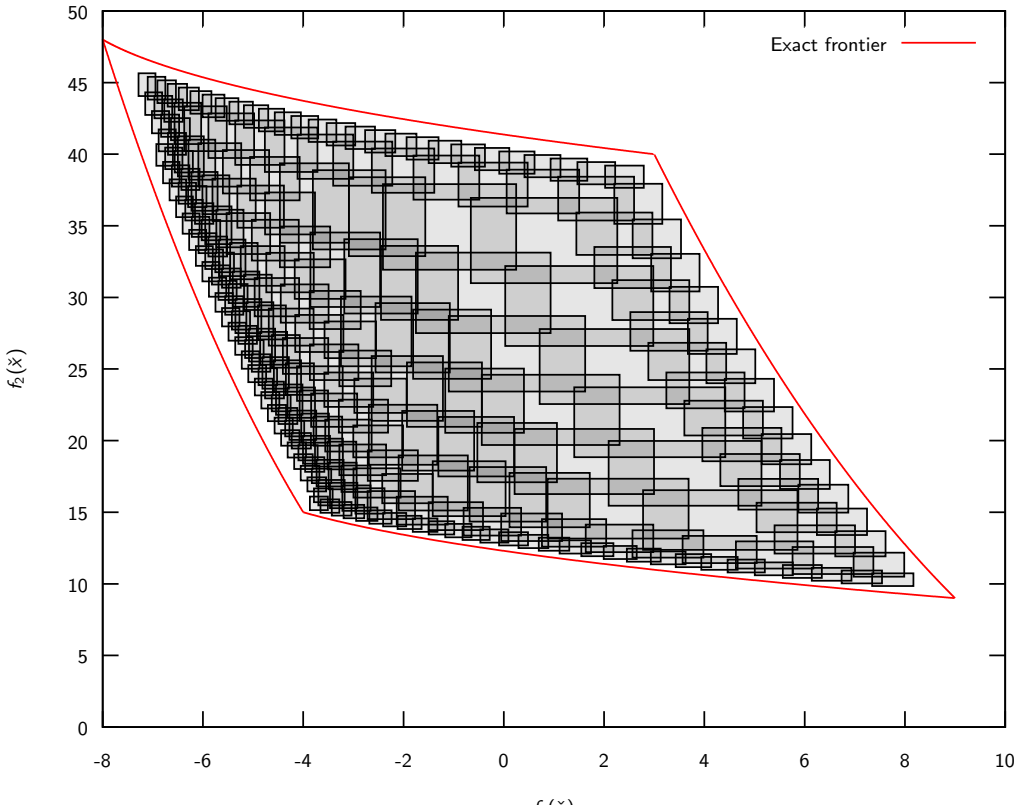
Characterization of the joint inner range of order 1 affine vectors: example

Example

Let $x = (x_1, x_2) \in [2, 3] \times [3, 4]$ and

$$f(x) = \begin{pmatrix} x_1^3 - 2x_1x_2 \\ x_2^3 - 2x_1x_2 \end{pmatrix}$$

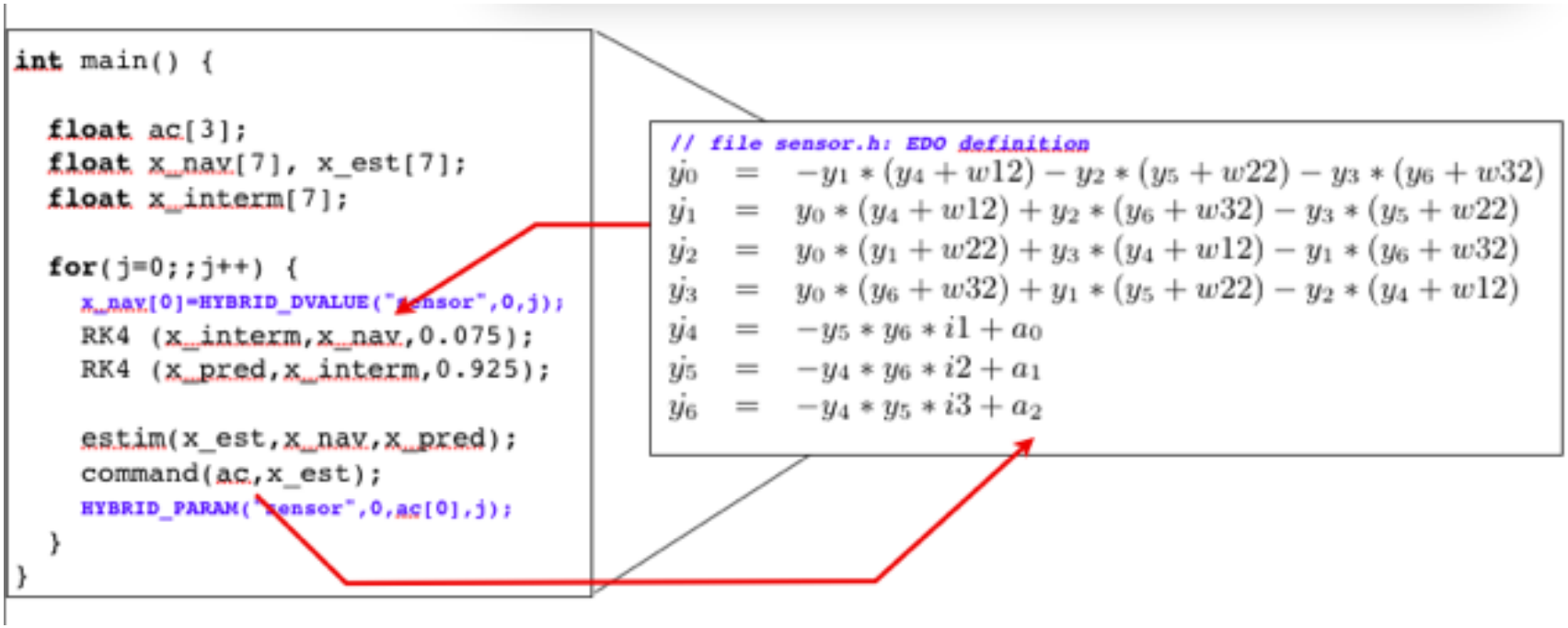
Joint inner range of the corresponding order 1 affine vectors costly but rarely needed



As an extension of classical program (discrete-time) analysis

- Classical program analysis: inputs given in ranges, possibly with bounds on the gradient between two values
  - Behaviour is often not realistic
- Hybrid systems analysis: analyze both physical environment and control software for better precision
  - Environment modelled by switched ODE systems
    - abstraction by guaranteed integration (the solver is guaranteed to outer-approximate the real solution)
  - Interaction between program and environment modelled by assertions in the program
    - sensor reads a variable value at time t from the environment,
    - actuator sends a variable value at time t to the environment,
- Other possible use of guaranteed integration in program analysis: bound method error of ODE solvers

Example: the ATV escape mechanism



- Time is controlled by the program (j)
- Program changes parameters (`HYBRID_PARAM`: actuators) or mode (not here) of the ODE system
- Program reads from the environment(`HYBRID_DVALUE`: sensors) by calling the ODE guaranteed solver

Could demonstrate convergence towards the safe escape state (CAV 2009, DASIA 2009 with Olivier Bouissou).

Temporal logics : general properties on trajectories

Linear time temporal logics

- Simple LTL (Pnueli 1979)
- Metric temporal logics (Koymans 1990) used for falsification of properties in real-time (hybrid) systems, (see e.g. Sankaranarayanan & Fainekos, HSCC 2012

Modalities

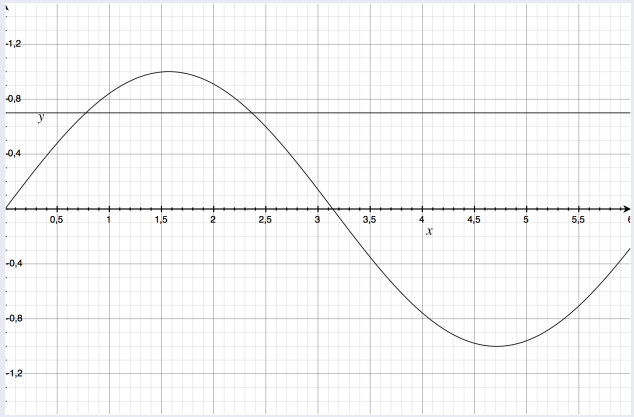
- 1st order predicate logics $\wedge$, $\vee$, $\Rightarrow$, $\neg$ etc.
- temporal modalities ; e.g.
  - $\Box\ p$: p is true always in the future
  - $\Diamond\ p$: p is going to be true at some point in the future

Adding time (MTL)

- We add a time interval as index to the modalities
- $\Box_{[0,300]}\ p$: p is always true for times (in the future) between 0 and 300
- $\Diamond_{[5,10]}\ p$: p is eventually true between times 5 and 10

Examples

This is a logics on states, for all trajectories



$\Diamond_{[0,2]}\ (y > 0.7)$ is true
but $\Box_{[0,2]}\ (y > 0.7)$ is false

Examples of interesting properties in control

- Stability (invariance) : $\Box_{[t,\infty[}(x \in K)$
- A signal x2 being close to a reference signal x1:
  $\Box_{[0,\infty[}\left((x1 > 0.7) \Rightarrow \Diamond_{[0,0.1]}(x2 > 0.7)\right)$ (whener x1 crosses threshold 0.7, so does x2 within $[0, 0.1]$ time unit - think of a decision taken by a system computing in finite precision)
- Bounded-time stabilization with respect to disturbances : whenever signal gets outside its reference range, it should be brought in this range in bounded time and remain there for some time
  $\Box_{[0,\infty[}\left((x \leq 100) \wedge ((x > 10) \rightarrow \Diamond_{[0,150]}\Box_{[0,10]}(x \leq 10))\right)$

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Quantifying on executions

Branching time logics

- CTL, CTL* (Clarke, Emerson 1981)
- Add quantification over paths : the future is not determined (as in e.g. dynamical systems with uncertainties, differential inclusions etc.)
- Intuitively : logical operators on paths, and on states

Syntactically

- A means "for all paths" in the future
- E means "exist a path" in the future
- This is combined with $\Box$ and $\Diamond$
- Example (reachability) : $E\Diamond\ p$, some state satisfying p is reachable in some future path
- Example (safety : p is invariant) : $A\Box\ p$, all states in all possible futures satisfy p

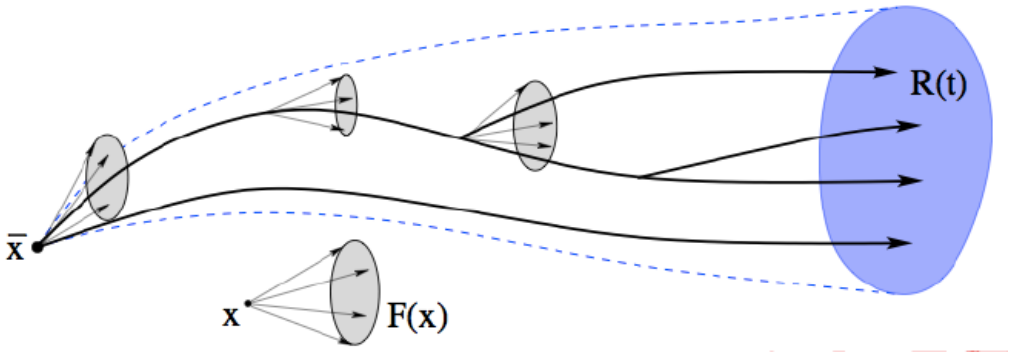
As used in e.g. UPPAAL

- Henzinger, Sifakis et al. 1992
- Several fragments and syntax, basically, add time constraints

Differential inclusions

Consider an uncertain dynamical system, e.g. a differential inclusion :

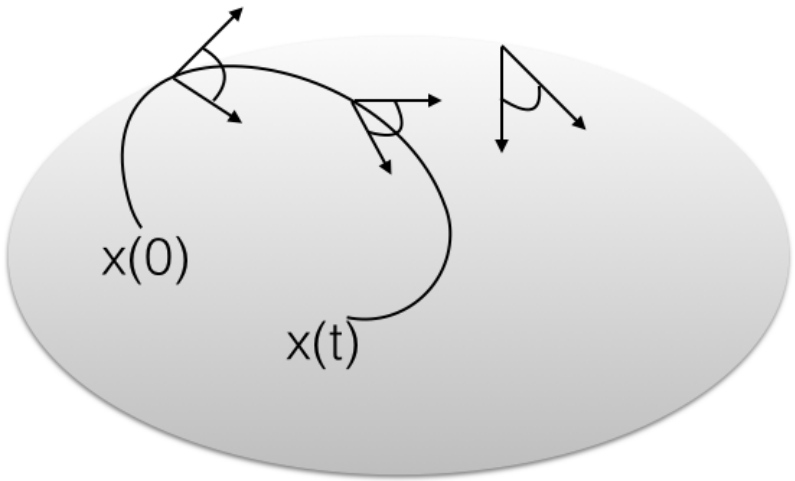
$$\dot{x} \in F(x)$$



Viability property

For all initial states in K, there exists a path such that in the future, $x \in K$ (the viability kernel) :

$$x(0) \in K \Rightarrow (E\Box\ (x \in K))$$



Semantics of TCTL

State formulas

- $\Box_I\ p$ is true (on state x) iff $\forall t \in I,\ p(t,x)$ is true
- $\Diamond_I\ p$ is true (on state x) iff $\exists t \in I,\ p(t,x)$ is true.
- Note that $\neg\Box_I\ p = \Diamond_I\ \neg p$

Path formulas

- $A\ p$ is true iff $\forall x,\ p(t,x)$ is true
- $E\ p$ is true iff $\exists x$ such that $p(x,t)$ is true
- Note also $\neg(A\ p) = E\ (\neg p)$

Combining modalities: examples

- $A\Box_I p$: $\forall x, \forall t \in I,\ p$ is true
- $A\Diamond_I p$: $\forall x, \exists t \in I$ (potentially depending on x then!), p is true
- $E\Box_I p$: $\exists x, \forall t \in I,\ p$ is true (x does not depend on t!)
- $E\Diamond_I p$: $\exists x, \exists t \in I,\ p$ is true

For now, we will only discuss these properties

Abstract model-checking

Abstraction

Now, we only have approximations of the trajectories $g(t, x_0)$, solution of the IVP $\dot{x} = f(x)$, $x(0) = x_0 \in \mathbf{x_0}$: suppose we have on time interval I an outer-approximation function $\overline{\mathbf{g}} : I \times \wp(\mathbb{R}^n) \to \wp(\mathbb{R}^n)$ and an inner-approximation function $\underline{\mathbf{g}} : I \times \wp(\mathbb{R}^n) \to \wp(\mathbb{R}^n)$ (on time $\times$ initial condition, generally in box $\mathbf{x_0}$), (semi-)decide some of TCTL formulas!

Most notable related work

- Combination of abstract interpretation and model-checking (Clarke, Grumberg & Long 1992) but not applied on numerical properties in general
- Falsification methods on MTL and hybrid systems (Sankaranarayanan & al. S-Taliro tool 2011) but only on one execution
- Monitoring, falsification, parameter synthesis of STL specs. on hybrid systems (Breach, Alexandre Donzé 2010)
- Monitoring of BLTL on hybrid systems (Goldsztejn et al. 2015)

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Interpreting TCTL

Sufficient conditions (proof)

- If p is true on $\overline{\mathbf{g}}(I, \mathbf{x_0})$ then $A\Box_I p$ is true (we will see later how to do this)
- If $\{x \in \mathbb{R}^n | p(x)\} \cap \underline{\mathbf{g}}(I, \mathbf{x_0}) \neq \emptyset$ then $E\Diamond_I p$ is true
- If $\exists t \in I$ such that p is true on $\overline{\mathbf{g}}(t, \mathbf{x_0})$ then $A\Diamond_I p$
- If p is true on $\underline{\mathbf{g}}(I, \mathbf{x_0})$ then $E\Box_I p$ (strong condition, we can do better, later)

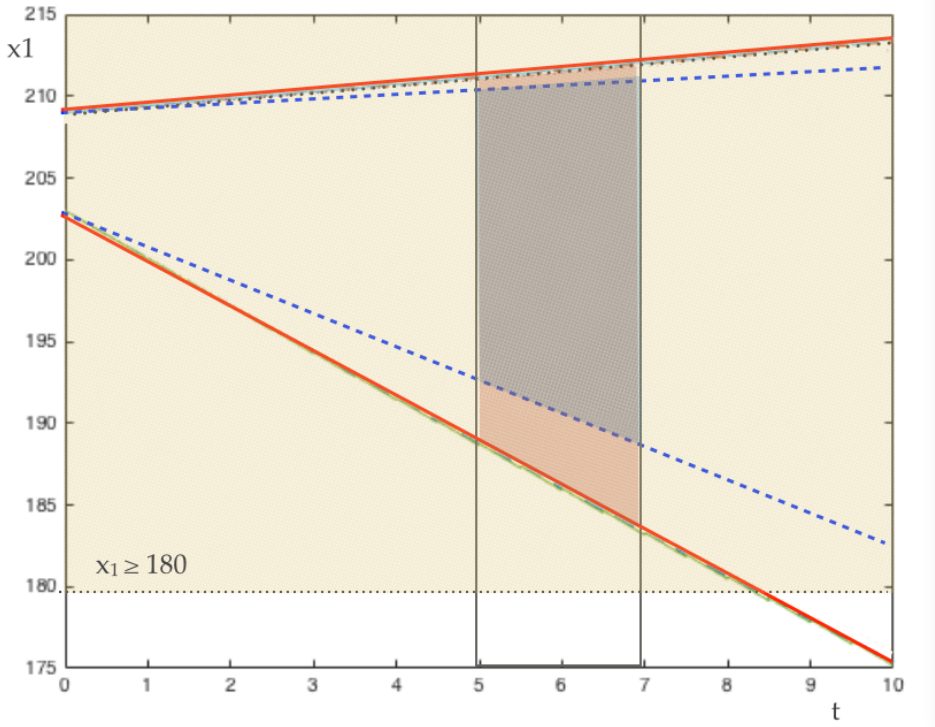
Necessary conditions (falsification)

- $E\Box_I p$ implies $\underline{\mathbf{g}}(I, \mathbf{x_0}) \cap \{x \in R^n | p(x)\} \neq \emptyset$
- $A\Diamond_I p$ implies $\overline{\mathbf{g}}(I, \mathbf{x_0}) \cap \{x \in R^n | p(x)\} \neq \emptyset$

Use of negation

- Use $\neg A\Box_I p = E\Diamond_I \neg p$; a sufficient condition on the latter (through the inner-approximation) implies that $A\Box p$ is false (whereas the interpretation using outer-approximation can only prove $A\Box p$ to be true)
- Similarly, we can use $\neg A\Diamond_I p = E\Box_I \neg p$

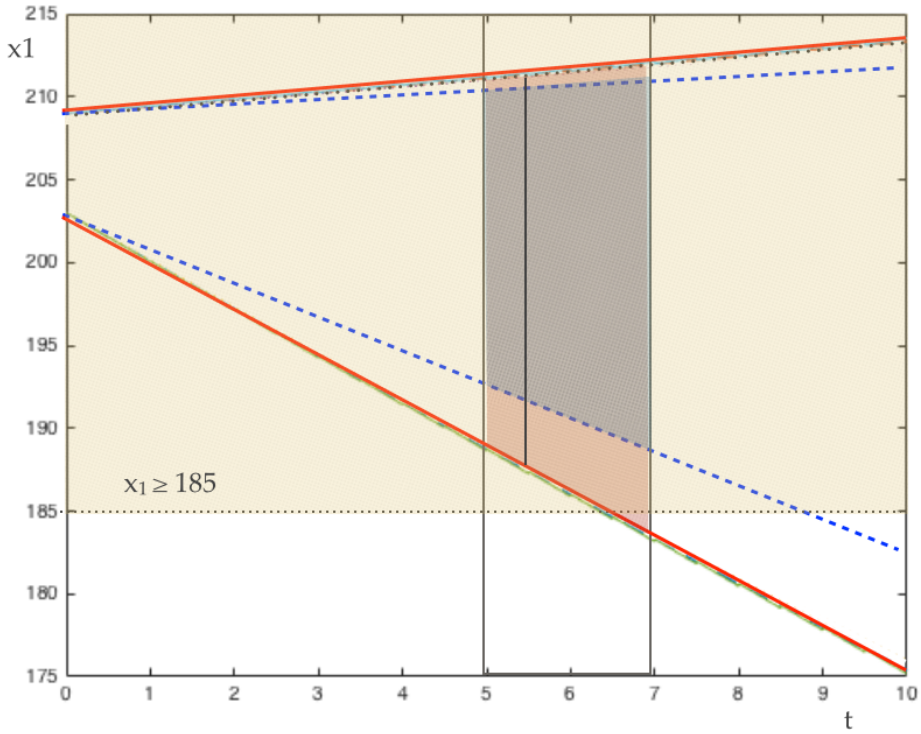
Interpreting TCTL



Sufficient conditions

- $A\Box_{I=[5,7]}(x_1 \geq 180)$ ($x \geq 180$ is true on $\overline{\mathbf{g}}(I, \mathbf{x_0})$)

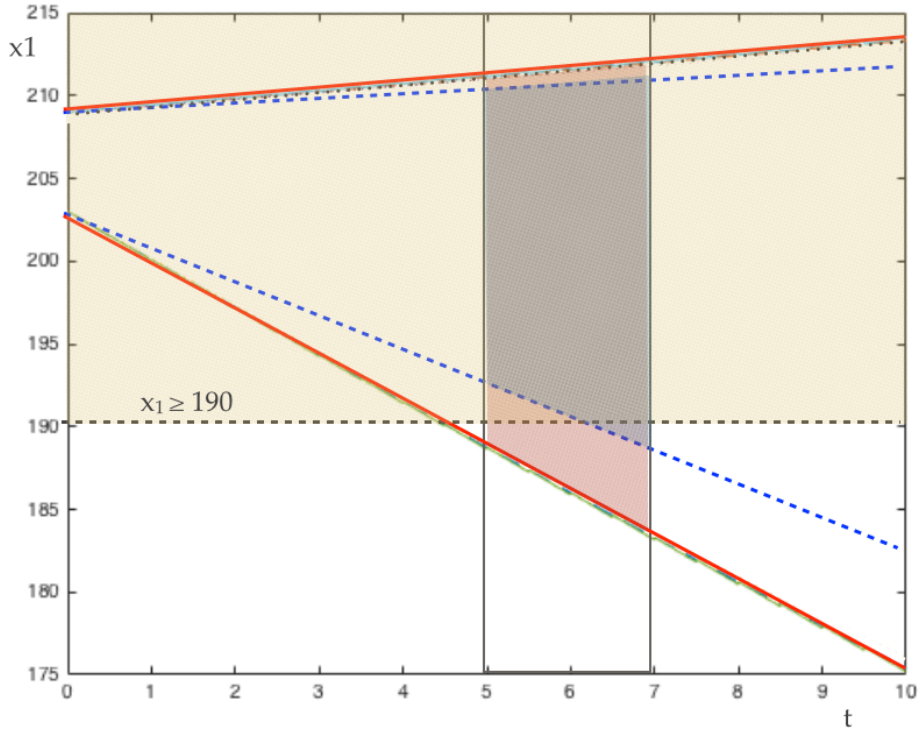
Interpreting TCTL



Sufficient conditions

- $A\Diamond_I(x_1 \geq 185)$ ($\exists t \in I$ such that p is true on $\overline{\mathbf{g}}(t, \mathbf{x_0})$)
- $E\Box_I(x_1 \geq 185)$ ($x_1 \geq 185$ is true on $\underline{\mathbf{g}}(I, \mathbf{x_0})$)
- We cannot prove $A\Box_I(x_1 \geq 185)$ is true, or false

Interpreting TCTL



Sufficient conditions

- $E\Diamond_I(x_1 \geq 190)$ $(\{x_1 \geq 190\} \cap \underline{\mathbf{g}}(I, \mathbf{x_0}) \neq \emptyset)$

Use of negation to falsify $A\Box_I(x_1 \geq 190)$

- Use $\neg A\Box_I(x_1 \geq 190) = E\Diamond_I(x_1 < 190)$
- $E\Diamond_I(x_1 < 190)$ $(\{x_1 < 190\} \cap \underline{\mathbf{g}}(I, \mathbf{x_0}) \neq \emptyset)$ hence $A\Box_I(x_1 \geq 190)$ is false

Problem statement (ODE)

- For ODE
$$\dot{x} = f(x) \text{ with } f : \mathbb{R}^n \to \mathbb{R}^n$$
- Suppose it has a unique solution on time interval $[0, T]$, for an initial condition x_0 at time 0.
- Suppose $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the C^1 function such that $t \to g(t, x_0)$ is the solution to this equation with initial condition $x_0 \in \mathbb{R}^n$.

Outer-approximation

- Use Taylor method (Moore, Berz & Makino etc.) for outer-approximating solution g to the ODE at some order m :

$$g_j(t, x_0) \quad = \quad x_{0,j} + \sum_{i=1}^{m} \frac{f_j^{(i)}(0,x_0)}{i!} t^i + \frac{f_j^{(m+1)}(\xi,x_0))}{(m+1)!} t^{m+1}$$

where $f^{(i)}$ is defined inductively as follows :

$$f_j^{(l+1)}(t, x_0) \quad = \quad \sum_{i=1}^{n} \frac{\partial f_j^{(l)}}{\partial x_i} f_i(g(t, x_0))$$

Bounding the remainder: Picard-Lindelöf iteration

Integral operator

- Let $F(g)(t,x) = x_0 + \int_0^t f(g(s,x))ds$
- Under simple hypotheses, the fixed point of F (on a small time interval $[0, T]$) exists and can be computed by iteration of F, and is the solution to our ODE

Rough enclosures

- Can also be used also to estimate the remainder $\frac{f_j^{(m+1)}(\xi,x_0)}{(m+1)!}$ $(\xi \in [0, T])$ since this depends on $g(\xi, x_0)$
- Suppose we have $\mathbf{x}$, an interval such that

$$\mathbf{x_0} + [0, T]f(\mathbf{x}) \subseteq \mathbf{x}$$

then $[0, T] \times \mathbf{x}$ contains all points $g(t, x_0)$, where g is a solution to our ODE on $[0, T]$, $t \in [0, T]$ and $x_0 \in \mathbf{x_0}$.

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For inner-approximations

- To compute inner-approximation, we need outer-approximations of the Jacobian of the solution, with respect to initial conditions, as in the discrete case
- The variational equation (as in e.g. Wilczak) is the ODE that is satisfied by g and its Jacobian $J^g_{j,i} = \frac{\partial g_j}{\partial x_{0,i}}$ with respect to the initial condition x_0 :

$$\begin{aligned} \frac{dg}{dt}(t, x_0) &= f(g(t, x_0)) \\ \frac{dJ^g_{j,i}}{dt}(t, x_0) &= \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k}(g(t, x_0)) J^g_{k,i}(t, x_0) \end{aligned}$$

Furthermore, the initial condition that g_j and $J^g_{j,i}$ satisfy are :

$$\begin{aligned} g_j(0, x_0) &= x_{0,j} \\ J^g_{j,i}(0, x_0) &= \delta_{i,j} \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker symbol.

Example

A simple ODE with uncertain initial values

- Consider the ODE $\dot{x} = x$ with $x_0 \in [0, 1]$ and $t \in \left[0, \frac{1}{2}\right]$
- The variational equation associated to this ODE is (noting $J^g = J^g_{1,1}$) :

$$\begin{array}{rcl} \frac{dg}{dt}(t, x_0) & = & g(t, x_0) \\ \frac{dJ^g}{dt}(t, x_0) & = & J^g(t, x_0) \end{array}$$

Furthermore, the initial condition that g and J^g satisfy are :

$$\begin{array}{rcl} g(0, x_0) & = & x_0 \\ J^g(0, x_0) & = & 1 \end{array}$$

Rough enclosures

- We see that $\mathbf{x} = [0, 2]$ satisfies $[0, 1] + \left[0, \frac{1}{2}\right][0, 2] \subseteq [0, 2]$ hence for all $t \in \left[0, \frac{1}{2}\right]$, for all $x_0 \in [0, 1]$, $g(t, x_0) \in [0, 2]$.
- Furthermore, we see that $\mathbf{x} = [1, 2]$ satisfies $1 + \left[0, \frac{1}{2}\right][1, 2] \subseteq [1, 2]$ so for all $t \in \left[0, \frac{1}{2}\right]$, for all $x_0 \in [0, 1]$, $J^g(t, x_0) \in [1, 2]$.

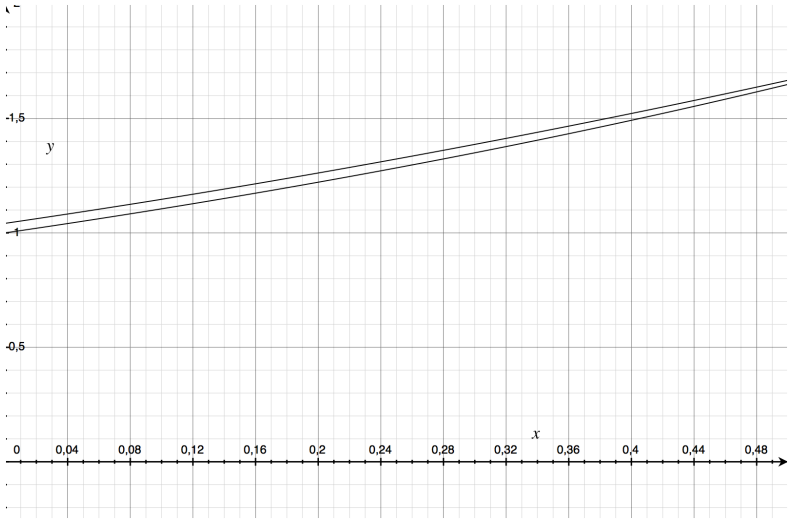
Outer-approximation for the solutions (order 3)

Taylor model

$$g(t, x_0) \quad = \quad x_0 + x_0 t + \frac{x_0}{2} t^2 + \frac{g(\xi, x_0)}{6} t^3$$

on $x_0 \in \mathbf{x_0} = \frac{1}{2} + \frac{1}{2}\varepsilon_1 \in [0, 1]$:

$$\overline{\mathbf{g}}(t, \mathbf{x_0}) \quad = \quad \left(\frac{1}{2} + \frac{1}{2}\epsilon_1\right)\left(1 + t + \frac{t^2}{2}\right) + \frac{[0,2]}{6} t^3$$



($\overline{g}$ and $g = exp$, over time)

For instance, at time $\frac{1}{2}$

$$\overline{\mathbf{g}}\left(\frac{1}{2}, \mathbf{x_0}\right) \quad = \quad \left[\frac{13}{16}, \frac{41}{48}\right] + \frac{13}{16}\epsilon_1$$

Hence $[0, e^{\frac{1}{2}}] \sim [0, 1.64872] \subseteq \overline{\mathbf{g}}\left(\frac{1}{2}, \mathbf{x_0}\right) = \left[0, \frac{5}{3}\right] \sim [0, 1.66667]$

Outer-approximation of the Jacobian (order 3), appli to inner-approx

Taylor model for the Jacobian

- We can outer-approximate the Jacobian, for all $t \in \left[0, \frac{1}{2}\right]$:

$$\begin{array}{rcl} J^g(t, x_0) & = & 1 + t + \frac{t^2}{2} + \frac{J^g(\xi, x_0)}{6} t^3 \\ \overline{\mathbf{J^g}}(t, \mathbf{x_0}) & = & 1 + t + \frac{t^2}{2} + \frac{[1,2]}{6} t^3 \end{array}$$

Example : inner-approximation at time $t = \frac{1}{2}$

- $\overline{\mathbf{J^g}}\left(\frac{1}{2}, \mathbf{x_0}\right) \quad \in \quad \left[\frac{79}{48}, \frac{5}{3}\right]$
- Mean value theorem, evaluated in $\varepsilon_1 = 0$, that is $x_0 = \text{mid}(\mathbf{x_0}) = \frac{1}{2}$, at time $t = \frac{1}{2}$, yields an inner-approximation of $\{g(\frac{1}{2}, x_0), x_0 \in \mathbf{x_0} = [0, 1]\}$:

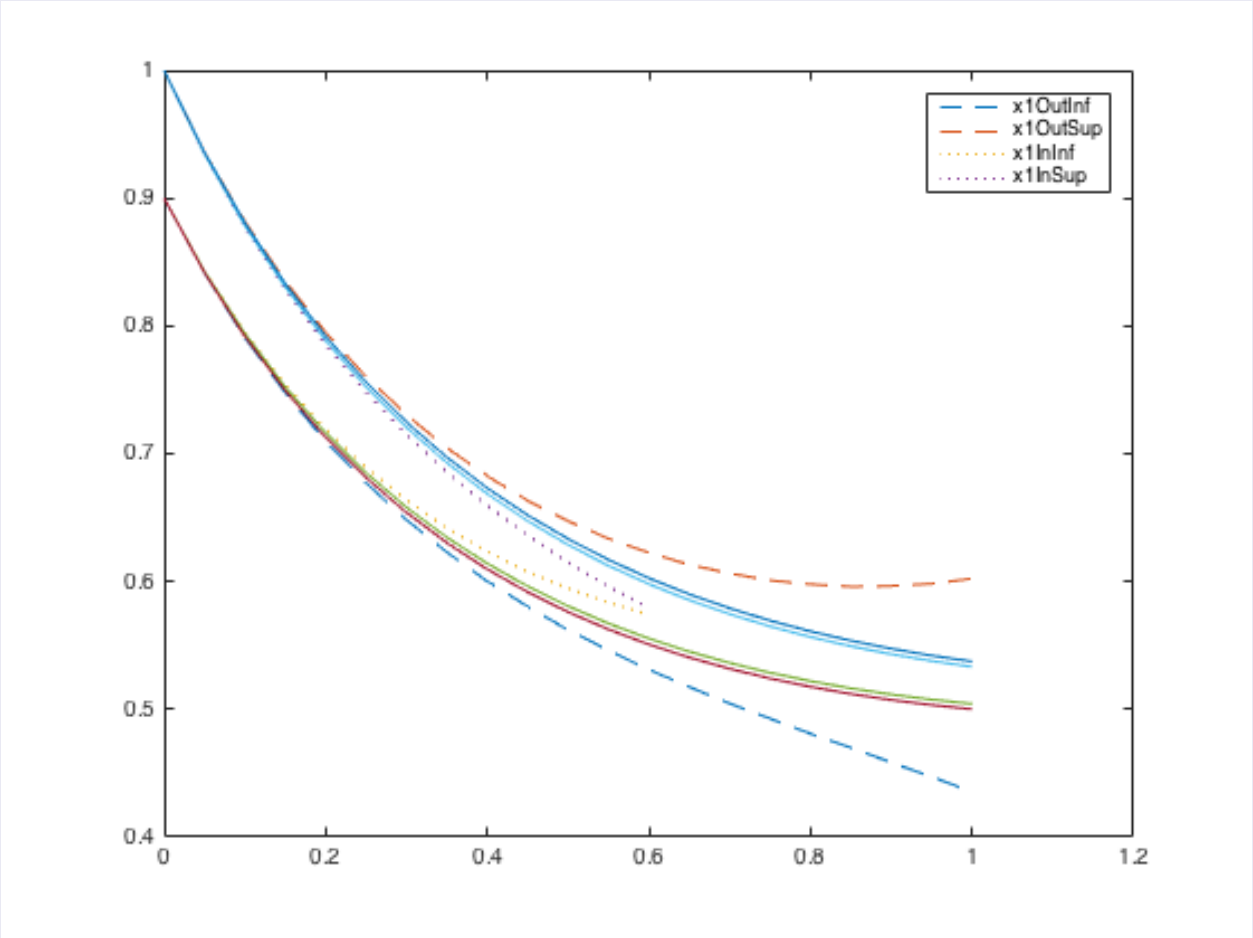
$$\begin{aligned} \underline{\mathbf{g}}\left(\tfrac{1}{2}, \mathbf{x_0}\right) &= \text{pro}\left(\overline{\mathbf{g}}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + \tfrac{1}{2}\overline{\mathbf{J^g}}\left(\tfrac{1}{2}, \mathbf{x_0}\right)[1, -1]\right) \\ &= \text{pro}\left(\underbrace{\left[\frac{13}{16}, \frac{41}{48}\right]}_{\text{proper}} + \tfrac{1}{2}\underbrace{\left[\frac{79}{48}, -\frac{79}{48}\right]}_{\text{improper}}\right) = \text{pro}\left(\underbrace{\left[\frac{157}{96}, \frac{3}{96}\right]}_{\text{improper}}\right) \\ &= \left[\tfrac{3}{96}, \tfrac{157}{96}\right] \sim [0.03, 1.635] \\ &\subseteq \left\{g\left(\tfrac{1}{2}, x_0\right), x_0 \in \mathbf{x_0}\right\} = \left[0, e^{\frac{1}{2}}\right] \sim [0, 1.649] \end{aligned}$$

Quality of the approximations: example of the Brusselator

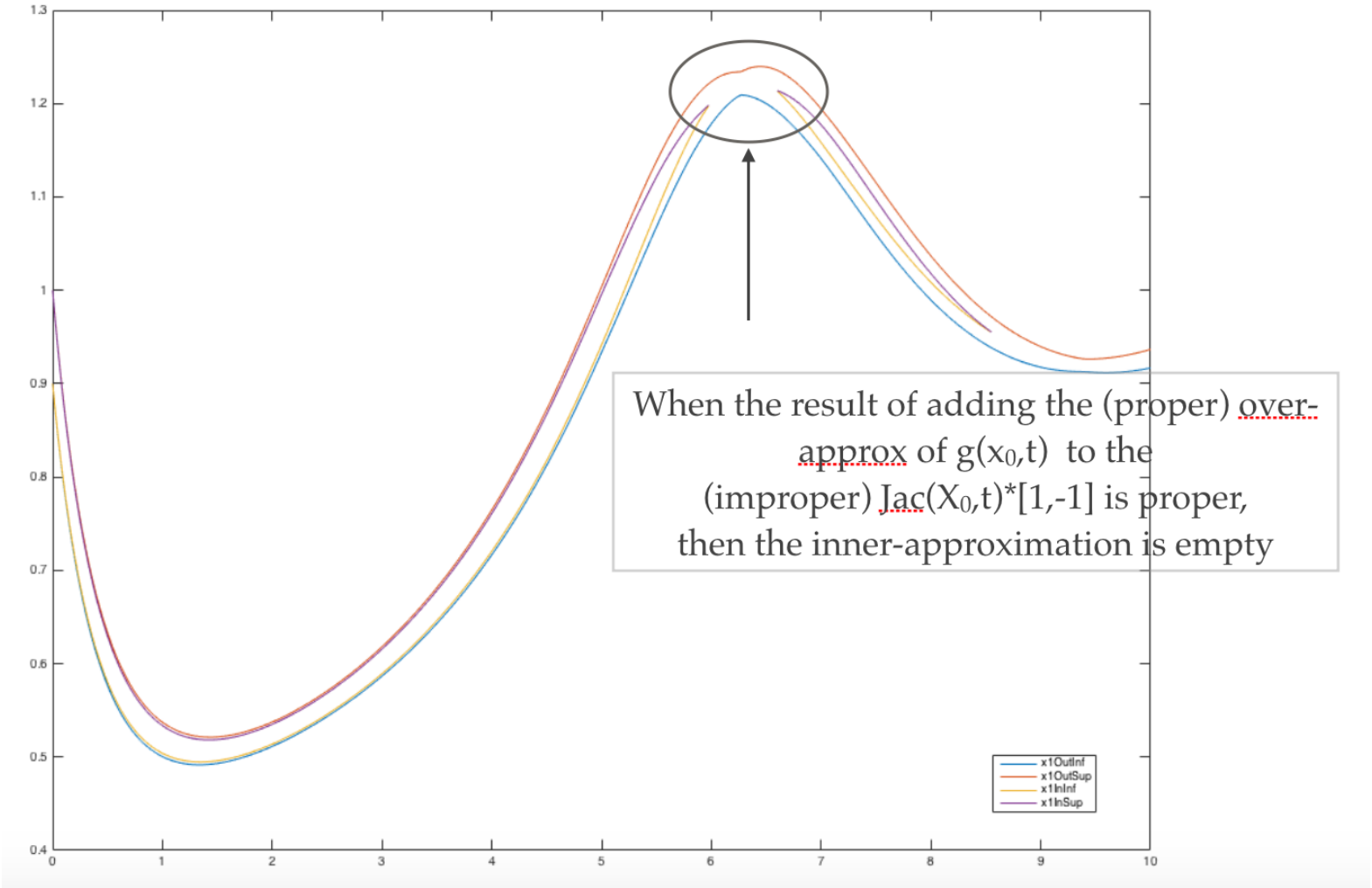
$$\begin{cases} \dot{x}_1 &= 1 + x_1^2 x_2 - 2.5 x_1 \\ \dot{x}_2 &= 1.5 x_1 - x_1^2 x_2 \end{cases}$$

with $x_1(0) \in [0.9, 1]$ and $x_2(0) \in [0, 0.1]$.

Taylor model of order 3 in t, interval vs affine arithmetic evaluation

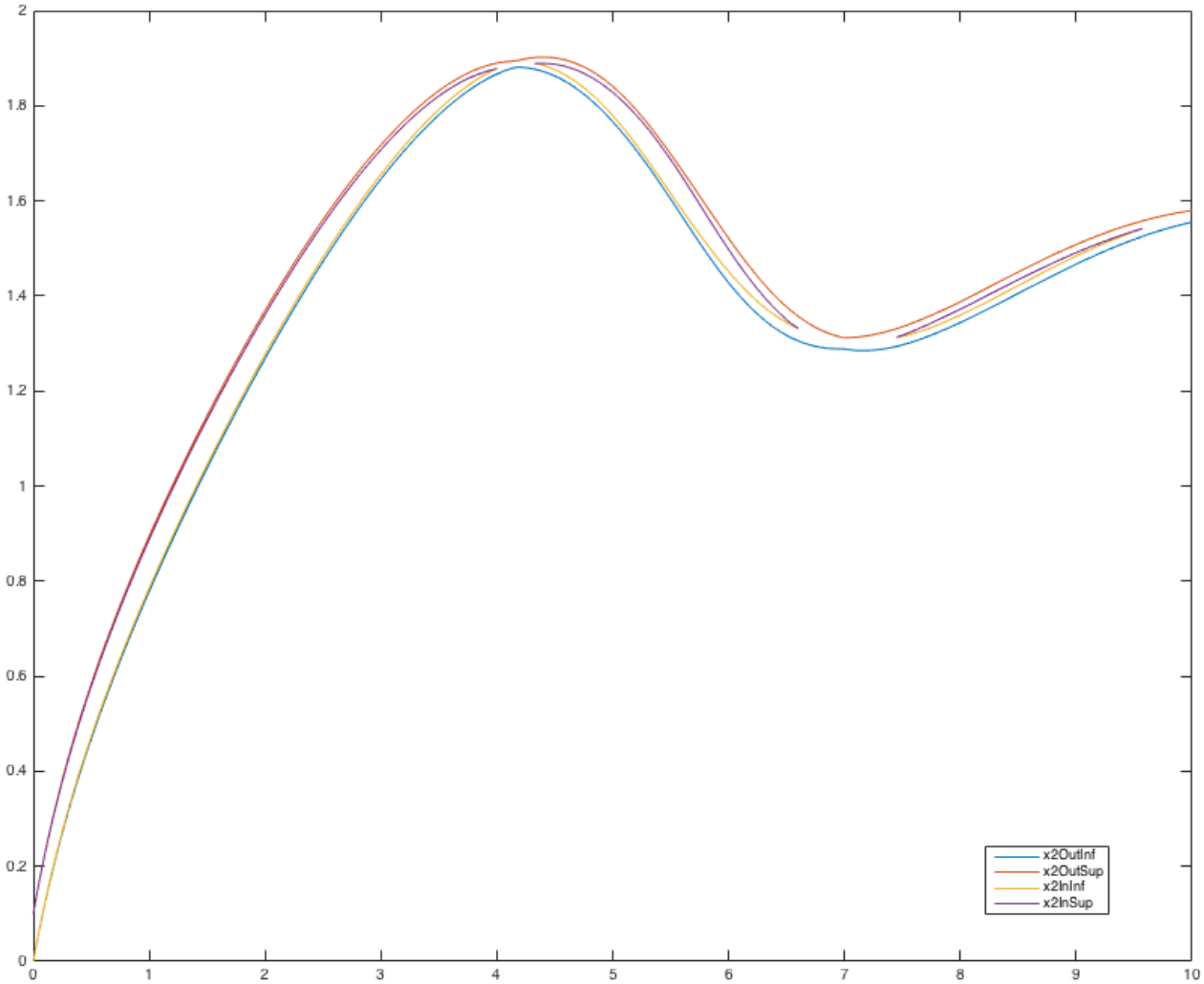


The Brusselator (x1): Taylor model of order 4 in t, up to t=10



When the result of adding the (proper) over-approx of $g(x_0,t)$ to the (improper) Jac(X_0,t)*[1,-1] is proper, then the inner-approximation is empty

The Brusselator (x2): Taylor model of order 4 in t, up to t=10



Back to inner-approximated function $\underline{\mathbf{g}}(t, \mathbf{x_0})$ for all $t \in [0, \frac{1}{2}]$ and TCTL

For this, we have to compute :

$$\overline{\mathbf{g}}(t, \text{mid}(\mathbf{x_0})) + \frac{1}{2}\overline{\mathbf{J^g}}(t, \mathbf{x_0})[1, -1] \subseteq \{g(t, x_0), x_0 \in \mathbf{x_0} = [0, 1]\}$$

So we need the center, for all t :

First,

$$\overline{\mathbf{g}}\left(t, \frac{1}{2}\right) = \left[\frac{1}{2} + \frac{1}{2}t + \frac{1}{4}t^2, \frac{1}{2} + \frac{1}{2}t + \frac{1}{4}t^2 + \frac{1}{3}t^3\right]$$

And the outer-approximation of the Jacobian, for all t and $x_0 \in \mathbf{x_0} = [0, 1]$:

$$\overline{\mathbf{J^g}}(t, \mathbf{x_0}) \in \left[1 + t + \frac{t^2}{2} + \frac{1}{6}t^3, 1 + t + \frac{t^2}{2} + \frac{1}{3}t^3\right]$$

Therefore

For all $t \in \left[0, \frac{1}{2}\right]$,

$$\underline{\mathbf{g}}(t, [0, 1]) = \left[\frac{t^3}{4}, 1 + t + \frac{t^2}{2} + \frac{t^3}{12}\right] \subseteq \{g(t, x_0), x_0 \in [0, 1]\}$$

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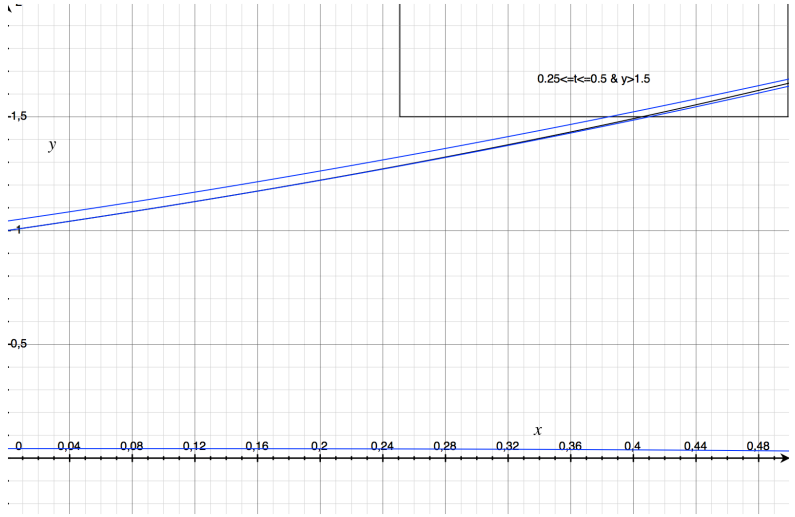
Function $\underline{\mathbf{g}}(t, x)$ and application



Application

- Consider $E\Diamond_{\left[\frac{1}{4},\frac{1}{2}\right]}\ (y > 1.5)$
- It will be true if $\underline{\mathbf{g}}\left(\left[\frac{1}{4},\frac{1}{2}\right],[0,1]\right)$ intersects $y > 1.5$
- We see that

$$\underline{\mathbf{g}}\left(\left[\frac{1}{4},\frac{1}{2}\right],[0,1]\right) = \left[\frac{1}{64}, 1.5 + \frac{7}{48}\right] \cap [1.5, \infty[\neq \emptyset$$

In general

Algorithmically

- We produce Taylor models for solutions and Jacobians (between all $[kT, (k+1)T]$)
- We deduce on all these time intervals $\overline{\mathbf{g}}$ and $\underline{\mathbf{g}}$ which are polynomial in t with coefficients in affine forms (linking them to uncertain initial values and parameters)
- For predicates $p = (f(t,x) \geq 0)$ with f polynomial, deciding p is true on $\underline{\mathbf{g}}(I, \mathbf{x_0})$ (resp. $\overline{\mathbf{g}}(I, \mathbf{x_0})$) can be done by any interval method (direct evaluation, affine forms, Bernstein polynomials etc.)
- For such predicates, deciding non-emptyness of the intersection of $\{x \in \mathbb{R}^n | p(x)\}$ with $\underline{\mathbf{g}}(I, \mathbf{x_0})$ (resp. $\overline{\mathbf{g}}(I, \mathbf{x_0})$) is a constraint satisfaction problem

Symbolic expressions in t and the ε_i allow refinements: example of $E\Box_I p$

- A sufficient condition is p true on $\underline{\mathbf{g}}(I, \mathbf{x_0})$
- If $p = (f(t,x) \geq 0)$, a finer criterion is to check the non-emptyness of the constraint on ϵ_i : $f(I, \underline{\mathbf{g}}(I, x_0)) \geq 0$ (polynomial in ϵ_i)

Example : aircraft (Lygeros 2011 & Sankaranarayanan 2014)

Differential model

$$\begin{aligned}
\dot{x}_1 &= -\frac{S\rho B_0}{2m}x_1^2 - g\sin\left(\frac{\pi x_2}{180}\right) + \frac{u_1}{m} - \frac{S\rho}{2m}x_1^2(B_1 u_2 + B_2 u_2^2) \\
\dot{x}_2 &= \frac{S\rho C_0}{2m}x_1 - g\frac{\cos\left(\frac{\pi x_2}{180}\right)}{x_1} + \frac{S\rho C_1}{2m}x_1 u_2 \\
\dot{x}_3 &= x_1 \sin\left(\frac{\pi x_2}{180}\right)
\end{aligned}$$

with initial conditions $x_1(0) \in [200, 260]$, $x_2(0) \in [-10, 10]$, $x_3(0) \in [120, 150]$.

- x_1 : speed, x_2 : angle, x_3 : altitude
- the inputs $u_1 \in [0.1, 0.2]$ and $u_2 \in [0.1, 0.2]$ represent respectively the thrust and the angle of attack.
- Constants : $B_0 = 0.07351, B_1 = -0.0015, B_2 = 0.00061, C_0 = 0.1667, C_1 = 0.109, m = 74000, g = 9.81, S = 158, \rho = 0.3804$
- The model is correct for small angle approximation for u_2.

Typical temporal properties to be checked (Sankaranarayanan 2013)

$\neg(\Box_{[0.5,1.5]}a \wedge \Diamond_{[3,4]}b)$, $\neg(\Box_{[0,4]}a \wedge \Diamond_{[3.5,4]}d)$, $\neg\Diamond_{[1,3]}e)$, $\neg(\Diamond_{[0.5,1]}a \wedge \Box_{[3,4]}g)$, $\neg(\Box_{[0,5]}h$, $\neg(\Box_{[2,2.5]}(i_1 \wedge i_2)$ where a is $240 \leq x_1 \leq 250$, b is $230 \leq x_1 \leq 240$, d is $240 \leq x_1 \leq 240.1$, e is $x_1 \geq 260$, g is $270 \leq x_1 \leq 280$, h is $190 \leq x_1 \leq 210$, i_1 is $190 \leq x_1 \leq 200$, i_3 is $190 \leq x_3 \leq 200$.

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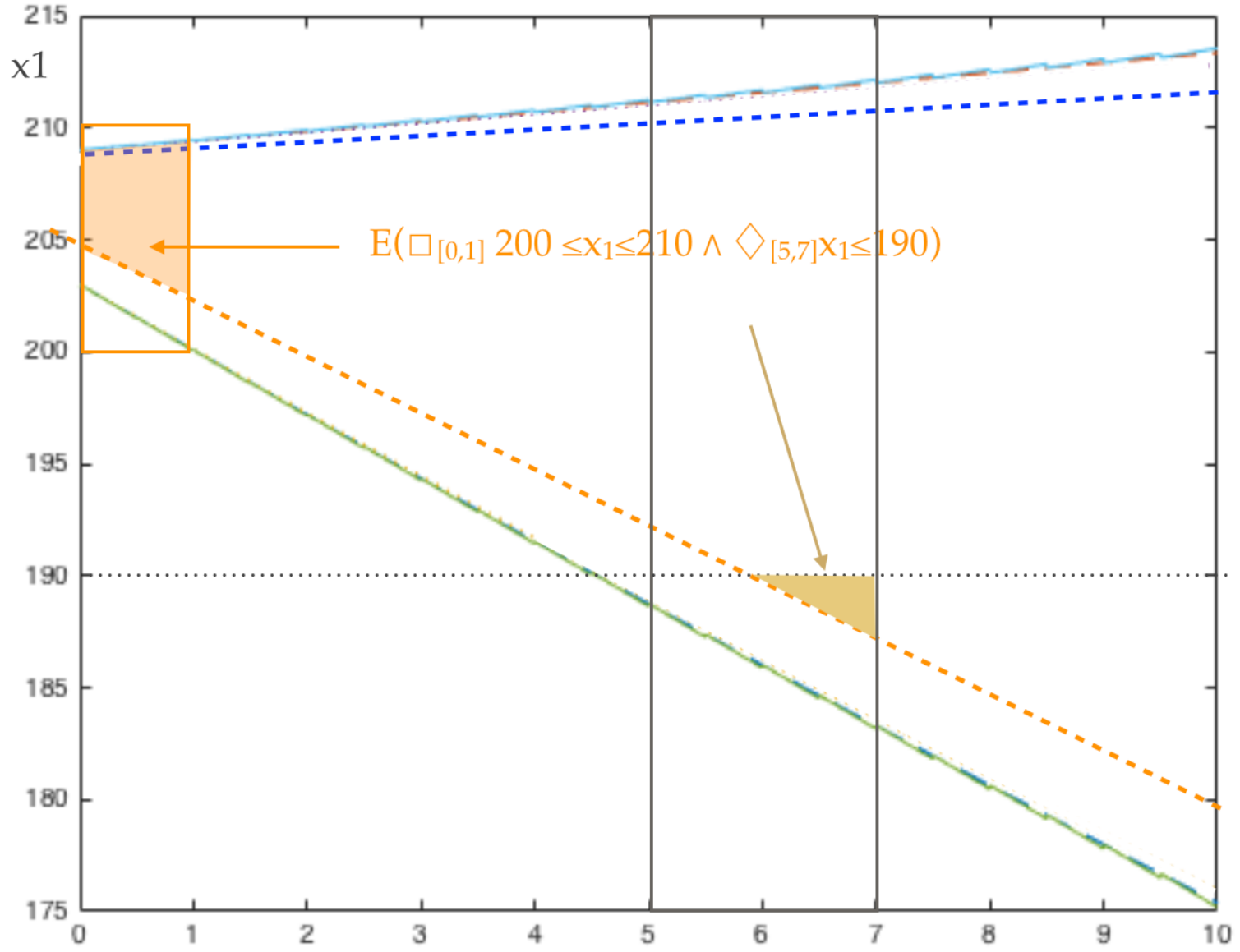
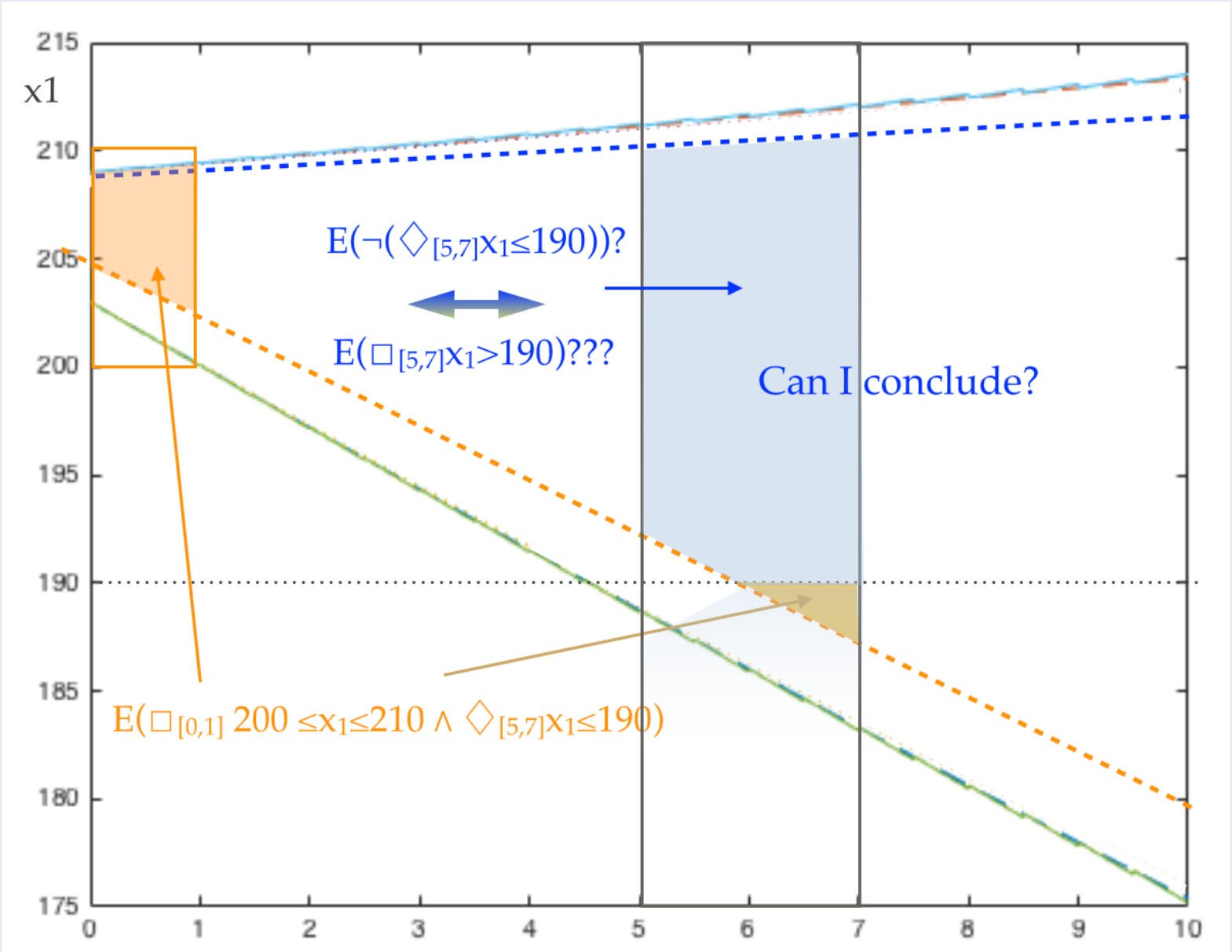
Example : aircraft (Lygeros 2011 & Sankaranarayanan 2014)

Inner-approximation: there exist trajectories for which $(\Box_{[0.,1.0]} a \wedge \Diamond_{[5,7]} b)$ is true



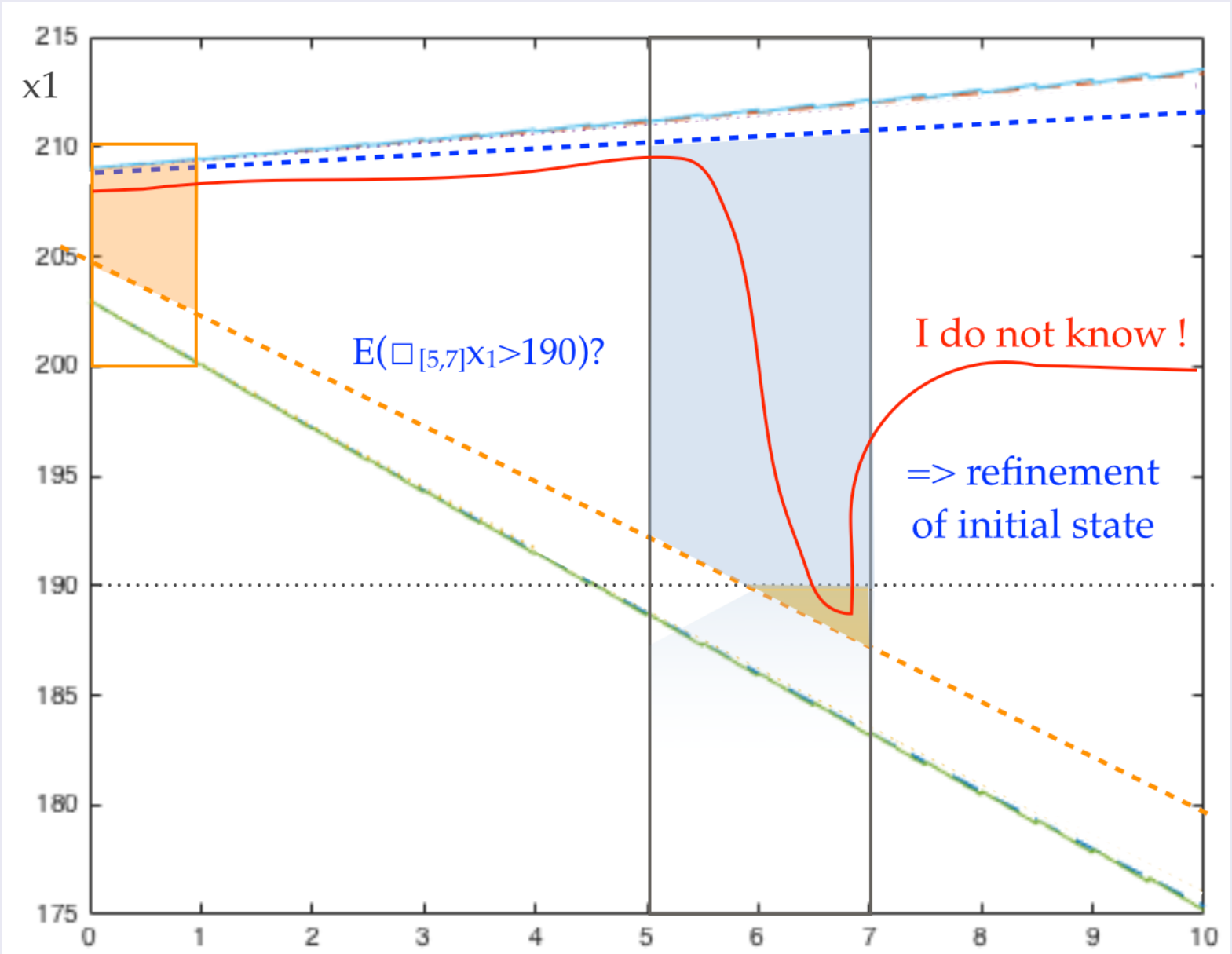
Example : aircraft (Lygeros 2011 & Sankaranarayanan 2014)

Does there also exist trajectories for which $(\Box_{[0.,1.0]} a \wedge \Diamond_{[5,7]} b)$ is false ?



Example : aircraft (Lygeros 2011 & Sankaranarayanan 2014)

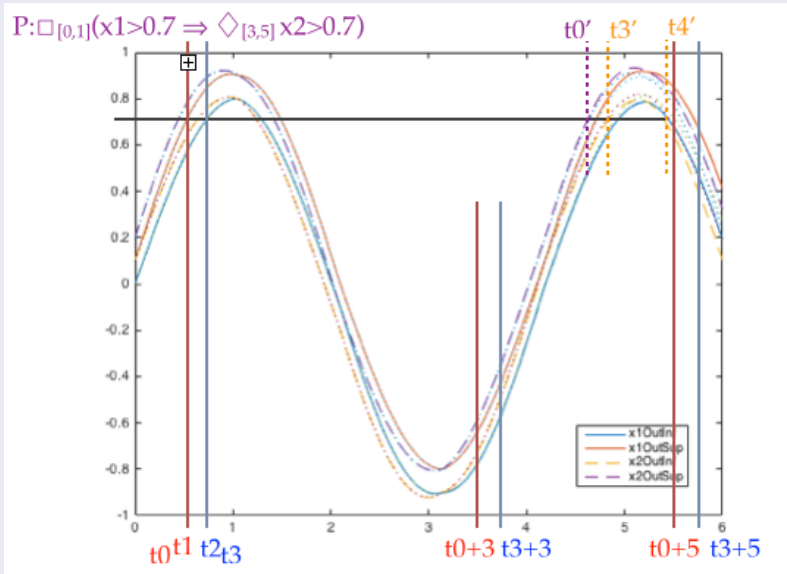
Does there also exist trajectories for which $(\Box_{[0.,1.0]} a \wedge \Diamond_{[5,7]} b)$ is false ?



Future work

Extension to the full logics

- Interpret the full fragment $A/E(stateformula)$ of TCTL
- Will need propagation of time constraints in the vein of (Ishii, Yonezaki, Goldsztejn 2015 - BLTL)



Any questions?