Validated integration of dissipative PDEs: chaos in the Kuramoto-Sivashinsky equations

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Trends in rigorous dynamics for PDEs

- BVP, equilibria, periodic orbits $\rightsquigarrow$ $\mathcal{F}(x) = 0$

  Arioli, Castelli, Figueras, Gameiro, James, Koch, Lessard, de la Llave, Nakao, Plum, Wanner,...

- Conley index, isolating segments

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- **Trajectory integration**

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Integration

- periodic orbits for ODEs, PDEs
- connecting orbits for ODEs, PDEs
- invariant manifolds
- bifurcations for ODEs

Global dynamics:

- chain recurrent sets
- Morse decomposition
- chaos for ODEs, PDEs
- existence and bifurcations of attractors

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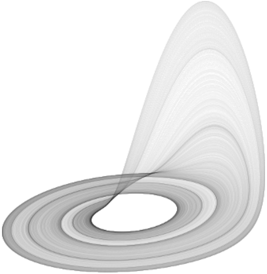
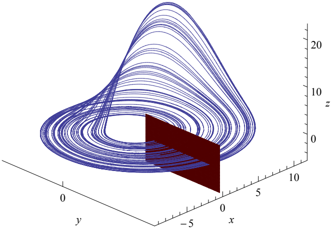
Global dynamics

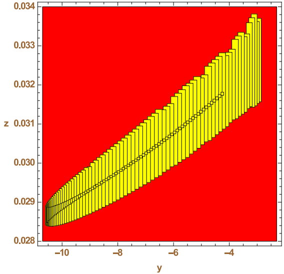
Example (Rössler system)

$$x' = -(y+z), \qquad y' = x + 0.2y, \qquad z' = 0.2 + z(x - 5.7)$$

There is a compact, connected invariant set which contains hyperbolic horseshoe.







CAPD library $(< 1 sec)$

Global dynamics

Uniformly hyperbolic chaotic attractor for a 4-dim ODE

$$\begin{cases} \dot{x} &= \omega_0 u, \\ \dot{u} &= -\omega_0 x + \left(A\cos(2\pi t/T) - x^2\right) u + (\varepsilon/\omega_0) y \cos(\omega_0 t), \\ \dot{y} &= 2\omega_0 v, \\ \dot{v} &= -2\omega_0 y + \left(-A\cos(2\pi t/T) - y^2\right) v + (\varepsilon/2\omega_0) x^2. \end{cases}$$

Parameters:

$$\omega_0 = 2\pi, \qquad A = 5, \qquad T = 6, \qquad \varepsilon = 0.5$$

DW, Uniformly hyperbolic attractor of the Smale-Williams type for a Poincaré map in the Kuznetsov system, SIAM J. App. Dyn. Sys. 2010, Vol. 9, 1263–1283.

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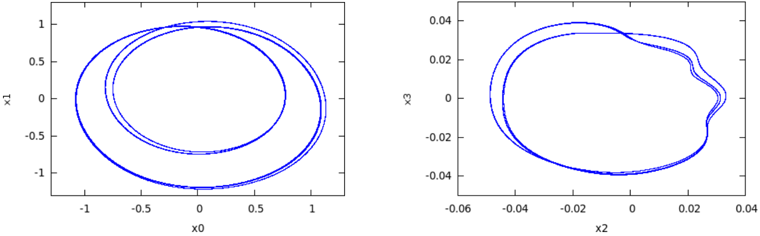

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Kuramoto-Sivashinsky equations

$$u_t = 2uu_x - u_{xx} - \nu u_{xxxx}$$

2π-periodic, odd

$$u(t,x) = -2\sum_{k=1}^{\infty} a_k(t)\sin(kx)$$

Infinite dimensional ODE

$$a_k' = k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{\infty} a_n a_{n+k}\right)$$

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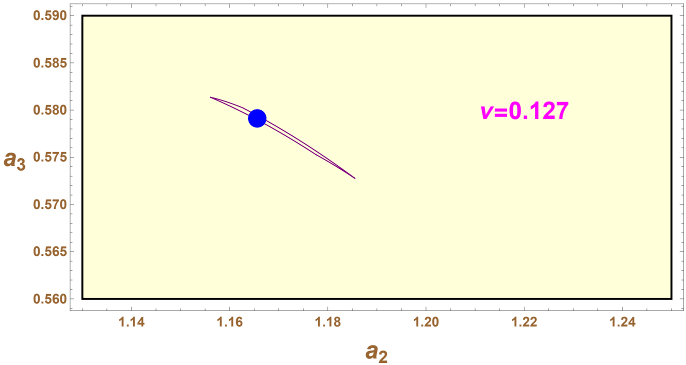
- ν - large $\Rightarrow u(x) \equiv 0$ is globally attracting
- $\nu = 0.127$ - nontrivial equilibria

  Zgliczyński & Mischaikow, FoCM’2001

- $\nu = 0.127 + [-1, 1] \cdot 10^{-7}$,
  $\nu = 0.125$,
  $\nu = 0.1215$,
  $\nu = 0.032$,
  (branches of) (symmetric) periodic orbits

  Zgliczyński, FoCM'2004, TMNA'2010



- $\nu = 4/150 \approx 0.02666\ldots$ - saddle hyperbolic periodic orbit
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- $\nu \in \{4/150, 0.02991, 0.0266, 0.111405\}$ - periodic orbits
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Strategy of the proof

1. Show that all (but finite number) of finite dimensional projections of the system are chaotic
2. invariant sets are compact
3. conclude the same for the full system

New tools:

- automatic differentiation for infinite dimensional systems
- validated integration of dPDEs

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M-dimensional Galerkin projection

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$\Pi_M = \{a_1 = 0 \wedge a_1' < 0\}$ - Poincaré section

$P_M : \Pi_M \to \Pi_M$ - Poincaré map

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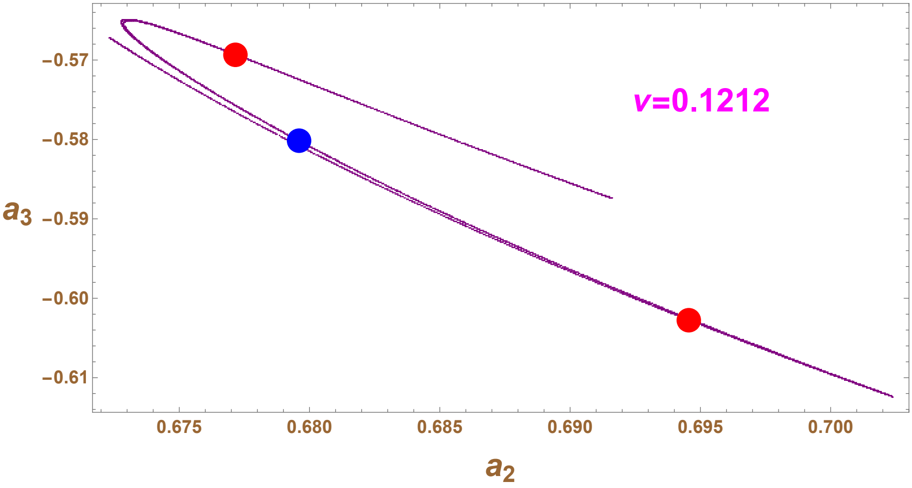
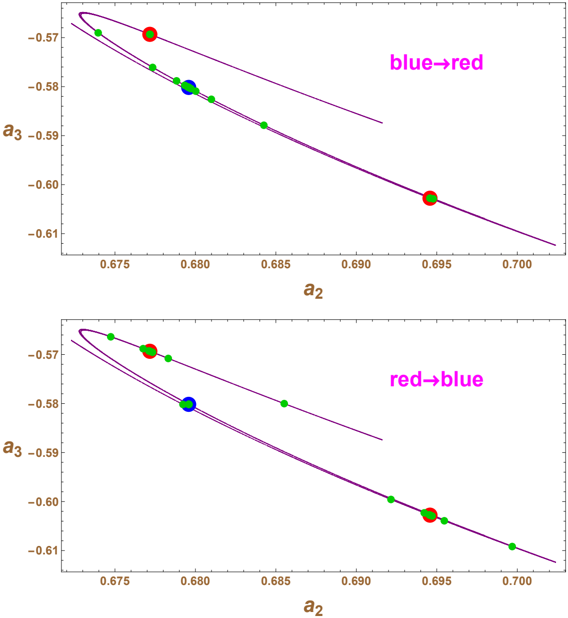
Observed chaotic attractor for P_M



Click here to run animation

Approximate heteroclinic orbits



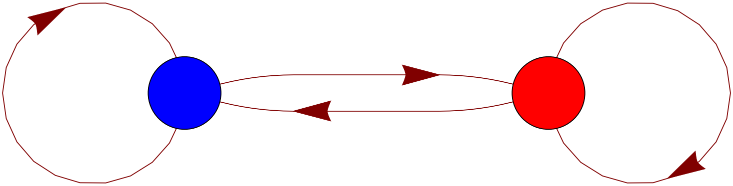
Result for Galerkin projections

Theorem

For $M \in \{12, 14, 16, 20, 25\}$ the M-dimensional Galerkin projection is chaotic:

- *There is an invariant set $\mathcal{H} \subset \Pi_M$ on which P_M is semiconjugated to a subshift of finite with positive topological entropy*
- *$\mathcal{H}$ contains countable infinity of periodic orbits: every periodic sequence of symbols is realized by a periodic orbit of P_M*

Graph of symbolic dynamics



- biinfinite path $\longrightarrow$ trajectory
- periodic path $\longrightarrow$ periodic orbit

Time of computation

M	wall time (64CPUs)
12	58 seconds
14	2.03 minutes
16	5.9 minutes
20	54.74 minutes
25	837 minutes

- **For all but $M = 12$ cases the same h-sets have been used.**
- CAPD standard ODE solver used – not optimized for stiff problems

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Main numerical result

Consider full infinite dimensional system:

$\Pi = \{a_1 = 0 \wedge a'_1 < 0\}$ $\qquad P : \Pi \to \Pi$

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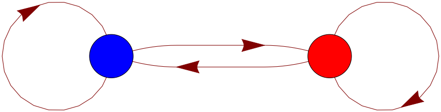

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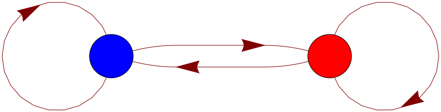

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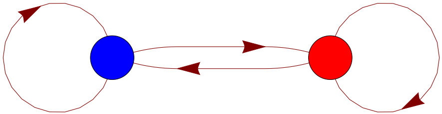

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Parameters of the algorithm

$m \leq M$ - positive integers (14,23)

Representation of sequences (`GeometricBound`)

- $k \leq m$ - doubleton, tripleton, any known from ODEs

$$(a_1, \ldots, a_m) \in x_0 + Cr_0 + Qr$$

- $k = m+1, \ldots, M$ - interval

$$a_k \in [a_k^-, a_k^+]$$

- $k > M$ - geometric decay (harmonic, mixed ...)

$$|a_k| \leq Cq^{-k}$$

with $q > 1$ and $C \geq 0$

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K-S equation in the Fourier basis

$$\begin{aligned} a'_k &= k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{\infty} a_n a_{n+k}\right) \\ &= L_k a_k + kE_k(a) + kI_k(a) \end{aligned}$$

E - finite part
I - infinite part

Each component is an univariate function

$$a_k(t) = \sum_{i=0}^{r} a_k^{[i]} t^i + [R_k].$$

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Automatic differentiation

$$(i+1)a_k^{[i+1]} = L_k a_k^{[i]} + kE_k^{[i]}(a) + kI_k^{[i]}(a)$$

$$= L_k a_k^{[i]} + kE_k^{[i]}(a) + F_k(a^{[0]}, a^{[1]}, \ldots, a^{[i]})$$

Technical lemma(s):

If

$$a^{[j]} = \texttt{GeometricBound}(C_j, q_j)$$

for $j = 0, 1, \ldots, i$ then

$$|F_k| \leq Dk^s \min\{q_j\}^{-k}$$

Lemma

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Variational equations

$$\frac{\partial a_k}{\partial a_c}(t) = a_{k,c}(t) = \sum_{i=0}^{\infty} a_{k,c}^{[i]} t^i.$$

Then

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Rough enclosure

$\dot{x} = f(x)$ - an ODE
X – set of initial conditions
$h > 0$ – time step

$$X(h) \subset \sum_{i=0}^{\infty} X^{[i]} h^i$$

$$X(h) \subset \sum_{i=0}^{p} X^{[i]} h^i + R$$

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$$X(h) \subset \sum_{i=0}^{\infty} X^{[i]} h^i$$

$$X(h) \subset \sum_{i=0}^{p} X^{[i]} h^i + R$$

Rough enclosure

$\dot{x} = f(x)$ - an ODE
X – set of initial conditions
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High-order enclosure

Theorem

If **Y** *is such that*

$$\sum_{i=0}^{p} X^{[i]}[0,h]^i + \mathbf{Y}^{[p+1]}[0,h]^{p+1} \subset \operatorname{int}(\mathbf{Y})$$

then for $t \in [0,h]$, $x \in X$ *there holds*

$$x(t) \in \mathbf{Y}$$

We can bound

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Important good prediction of h and **Y**

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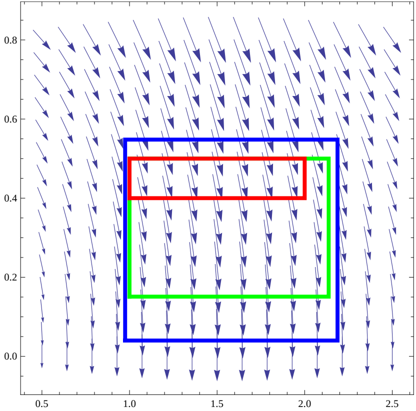
Important good prediction of h and $\mathbf{Y}$

Example

$$x'' = -\sin(x) + 0.1x', \qquad h = 0.25$$



$$[X] = [1,2] \times [0.4, 0.5]$$

$$[Y] = [X] + h[-.2, 1.5] * f([X]) \subset [0.9749, 2.1875] \times [0.04, 0.548]$$

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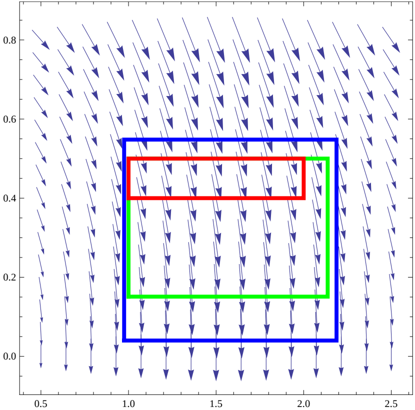

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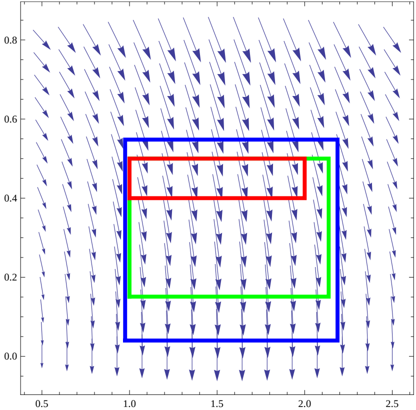

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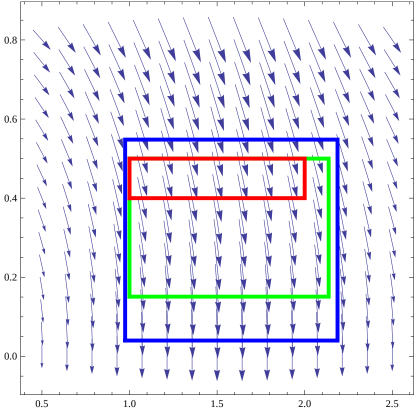

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Enclosure algorithm for PDEs

Parameters

- $p \geq 1$ - order of enclosure
- D - positive integer
  number of modes on which High–Order Enclosure acts
- h_0 - a candidate for the time step

Main steps

1. predict enclosure on $(a_1, \ldots, a_D)$ with h_0
2. compute enclosure for $k > D$ using **isolation**
3. validate enclosure on $(a_1, \ldots, a_D)$
   adjust final step $h \leq h_0$

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ε - user specified tolerance per one step

Predict a high–order enclosure for $k \leq D$

$$Y_k = \sum_{i=0}^{p} a_k^{[i]}[0, h_0]^i + [-\varepsilon, \varepsilon]$$

Conditional enclosure for $k > D$

$$Y = \texttt{GeometricBound}((Y_1, \ldots, Y_M), C, q)$$

- Assume $(Y_1, \ldots, Y_D)$ is an enclosure for $[0, h_0]$
- Enlarge $Y_{D+1}, \ldots, Y_M$ and C as long as vector field is pointing inwards the interval Y_k, $k > D$

$$\mathbf{L_k a_k} \in \Theta(\mathbf{k}^4)\mathbf{q}^{-\mathbf{k}}, \qquad \mathbf{L_k} < \mathbf{0}$$

$$\mathbf{N_k(a)} \in \mathbf{O}(\mathbf{k}^2)\mathbf{q}^{-\mathbf{k}}$$

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Validate $(Y_1, \ldots, Y_D)$

From prediction:

$$Y_k = \sum_{i=0}^{p} a_k^{[i]}[0, h_0]^i + [-\varepsilon, \varepsilon]$$

Check

$$Y_k^{[p+1]}[0, h_0]^{p+1} \subset [-\varepsilon, \varepsilon]$$

If not satisfied then find **h** $\leq h_0$ such that

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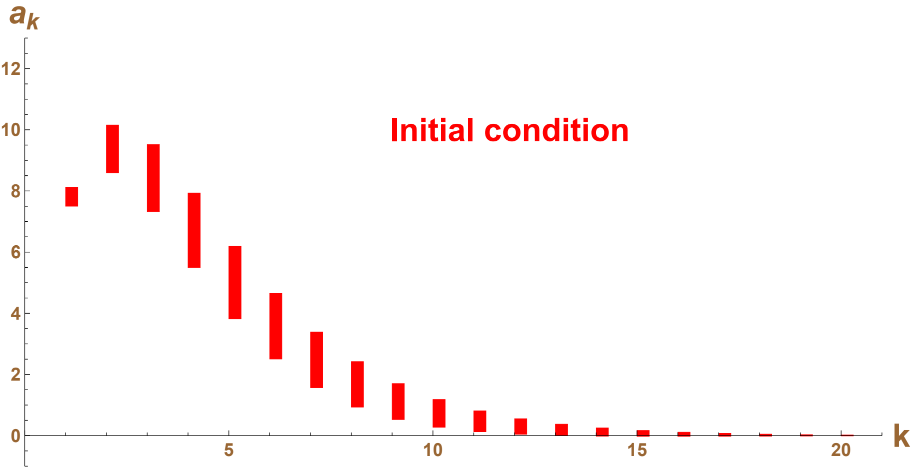
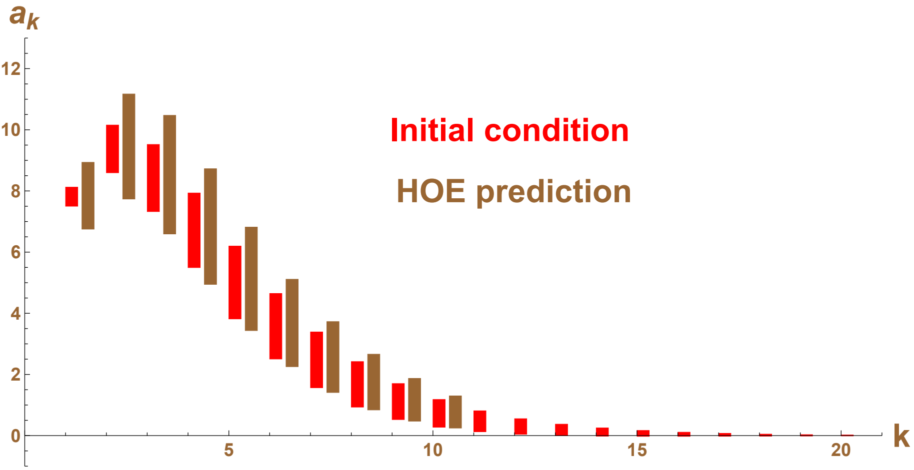


Adjust *h* if necessary



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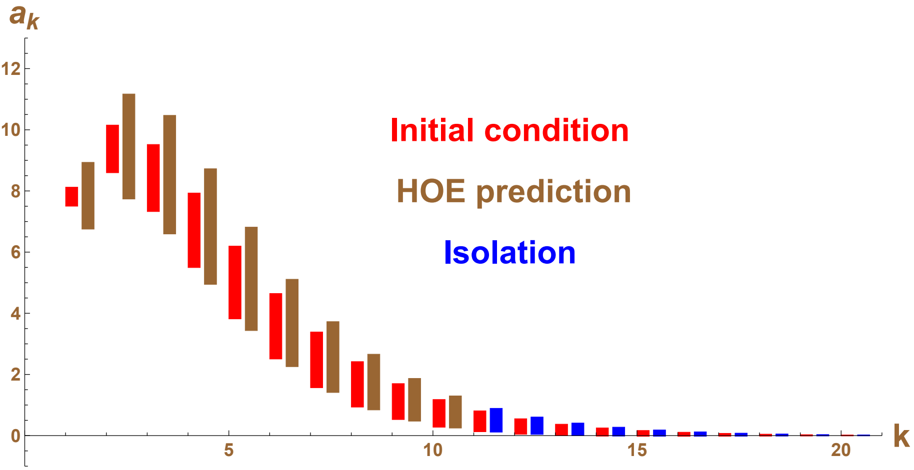

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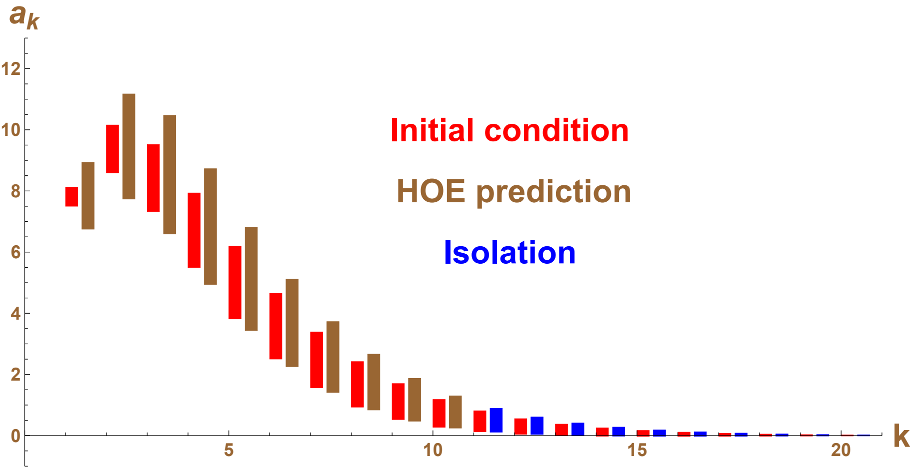

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Taylor method

$$a(h) = \sum_{i=0}^{p} a^{[i]} h^i + Y^{[p+1]}[0,h]^{p+1} = \Phi(h,a) + R.$$

Split initial condition

$$a = (a_1, \ldots, a_M, a_{M+1}, a_{M+2}, \ldots) = (x_0 + \Delta x, y)$$

One step enclosure for $k = 1, \ldots, M$:

$$\Phi_k(h,a) \subset \Phi_k(h,(x_0,y)) + D_x \Phi_k(h,a) \cdot (\Delta x, y)$$

One step enclosure for $j > M$:

$$Y_k' < L_k Y_k + C \quad \Rightarrow \quad a_k(h) < \left(Y_k(0) - \frac{C}{-L_k} \right) e^{hL_k} + \frac{C}{-L_k}$$

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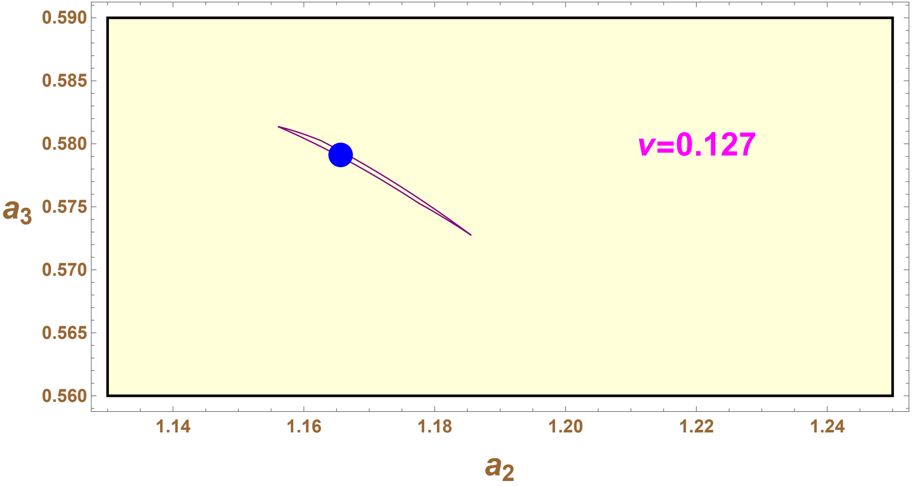
Important property of the algorithm:

The algorithm integrates simultaneously all N–dimensional Galerkin projections with $N > M$.

Proof of stable periodic orbit

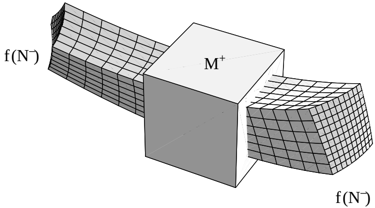
Result reproduced from Zgliczyński FoCM'2004



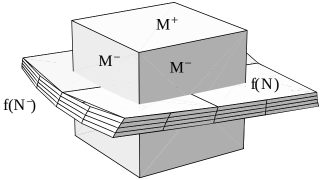
u_i	$P_i(u)$	λ_i
$[-1, 1] \cdot 10^{-5}$	$[-5.45, 5.45] \cdot 10^{-6}$	0.5258
$[-1, 1] \cdot 10^{-5}$	$[-9.85, 9.81] \cdot 10^{-7}$	0.0903
$[-1, 1] \cdot 10^{-5}$	$[-5.86, 4.67] \cdot 10^{-9}$	$3.5 \cdot 10^{-8}$
$[-1, 1] \cdot 10^{-5}$	$[-6.61, 4.32] \cdot 10^{-9}$	$1.65 \cdot 10^{-8}$
$[-1, 1] \cdot 10^{-5}$	$[-8.02, 5.65] \cdot 10^{-9}$	$-3.77 \cdot 10^{-9}$
$[-1, 1] \cdot 10^{-5}$	$[-6.62, 8.19] \cdot 10^{-9}$	$-4.01 \cdot 10^{-11}$
$[-1, 1] \cdot 10^{-5}$	$[-7.30, 9.62] \cdot 10^{-9}$	$-8.94 \cdot 10^{-10}$
$[-1, 1] \cdot 10^{-5}$	$[-2.15, 1.53] \cdot 10^{-9}$	$-6.69 \cdot 10^{-11}$
$\dots$	$\dots$	
$k > 23$	$k > 23$	
$10^{-5}(1.5)^{-k}$	$5.01 \cdot 10^{-8}(1.5)^{-k}$	

Covering relations

P. Zgliczyński, M. Gidea, JDE'2004



One unstable direction



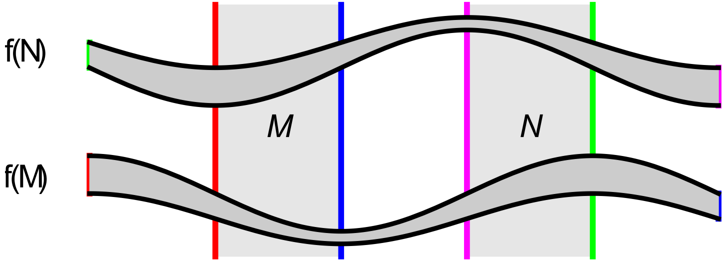
Two unstable directions

Theorem (Zgliczyński, Gidea)

Every binifinite (periodic) sequence of covering relations is realised by a (periodic) trajectory.

Model 2D situation:

- N covers M and itself
- M covers N and itself



- There are points jumping between N and M in any prescribed order $\{N, M\}^{\mathbb{Z}}$
- periodic sequence $\{N, M\}^{\mathbb{Z}} \rightsquigarrow$ periodic point

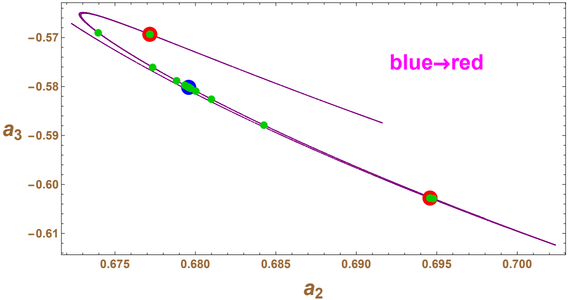
Chaos in the KS equations:

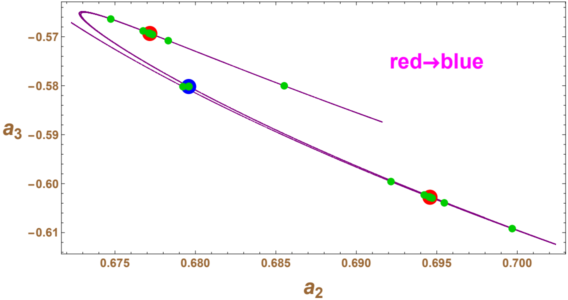
1. Find approximate two periodic orbits P_1 and P_2

2. find approximate finite trajectories
   (a) H_{12}: starting very close to P_1 and ending very close to P_2
   (b) H_{21}: starting very close to P_2 and ending very close to P_1

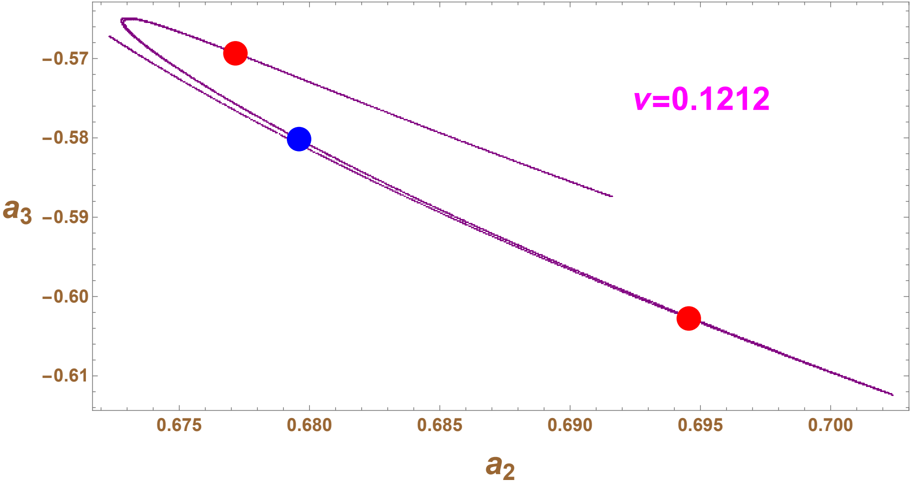
3. construct sequence of covering relations:
   (a) $P_1 \Longrightarrow P_1 \Longrightarrow H_{12}^1 \Longrightarrow \cdots \Longrightarrow H_{12}^n \Longrightarrow P_2$
   (b) $P_2 \Longrightarrow P_2 \Longrightarrow H_{21}^1 \cdots \Longrightarrow \cdots H_{21}^m \Longrightarrow P_1$

Approximate heteroclinic orbits





Unstable periodic orbit for $\nu = 0.1212$



Data from the proof of blue⟹blue

u_i	$P_i(u)$	λ_i
$3.8[-1,1]\cdot 10^{-6}$	$[-8.09, 8.09]\cdot 10^{-6}$	-1.7704
$1.9[-1,1]\cdot 10^{-7}$	$[-4.33, 4.59]\cdot 10^{-8}$	-0.06511
$1.9[-1,1]\cdot 10^{-7}$	$[-2.35, 1.68]\cdot 10^{-8}$	$-2.92\cdot 10^{-16}$
$1.9[-1,1]\cdot 10^{-7}$	$[-0.718, 1.13]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-0.982, 1.40]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-1.33, 2.04]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-2.86, 3.64]\cdot 10^{-9}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-2.75, 1.67]\cdot 10^{-9}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-3.64, 4.31]\cdot 10^{-10}$	≈ 0
$\ldots$	$\ldots$	
$k > 23$	$k > 23$	
$1.9\cdot 10^{-7}(1.5)^{-k}$	$2.64\cdot 10^{-9}(1.5)^{-k}$	

Thank you for your attention